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ABSTRACT

Appendix I is concerned with the structure of axiomatic theories. Both the syntactic and deductive structure exhibited by axiomatic theories is investigated. Dependency is defined and a method is described that can be used to identify all dependencies that exist between axiomatic components. The ability to eliminate defined concepts and non-creativity of definitions is discussed. A theorem that asserts that all defined concepts can be eliminated from an axiomatic theory is given. Appendix II contains the matrices for the photography example in Chapter I. (MK)

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FINAL TECHNICAL REPORT

VOLUME II

STRUCTURING AND CLASSIFYING
THE CONTENT OF INSTRUCTIONAL SUBJECT MATTER

(Appendices)

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APPENDIX I

The Structure of Axiomatic Theories

Preface

This paper is concerned exclusively with the structure of axiomatic theories. Both the syntactic and deductive structure exhibited by axiomatic theories is investigated. Dependency is defined and a method is described that can be used to identify all dependencies that exist between axiomatic components. Eliminability of defined concepts and non-creativity of definitions are discussed. The paper concludes with a theorem that asserts that all defined concepts are eliminable from an axiomatic theory.

Axiomatic methodology was first developed in ancient Greece approximately 2,300 years ago. As is well known Euclid applied the axiomatic method in order to systematize geometry. Since Euclid, axiomatic methodology has played an increasingly important role in mathematics and disciplines, such as physics, that rely heavily on mathematics. More recently the axiomatic process has been used within empirical disciplines such as biology, economics, sociology, psychology and anthropology. Normally these areas of science do not rely heavily on mathematics, and this utilization of axiomatics can be seen to be one way of introducing mathematics into these scientific investigations. Thus, a clear understanding of the axiomatic process is of considerable scientific importance.

Axiomatics is a method through which a theory of a particular subject matter may be developed. An axiomatic theory typically contains four elements: primitive concepts, defined concepts, axioms and theorems. The primitive concepts are taken to be intuitively comprehensible and are not defined within the theory. Normally, an effort is made to minimize the number of primitive concepts. Defined concepts are defined within the theory on the basis of primitive and other defined concepts. Concepts are defined in such a way that the definitions do not substantially strengthen the theory and, hence, defined concepts need not be introduced. Axioms are statements that are assumed to be true within the theory. In general, axioms provide the basic assumptions that determine the interrelationships between primitive concepts. Normally, the number of axioms is minimized with an effort being made to choose as axioms statements that are intuitively understandable and acceptable. Theorems are statements derived from the

axioms, definitions and other theorems. Thus, theorems are the statements which are held to be true of the subject matter under investigation. It can be seen from this brief description of axiomatics that the axiomatic method focuses on the development of theories for the investigation of particular subject matters.

Before continuing the analysis of axiomatics, some of the advantages of utilizing the axiomatic method will be described. Perhaps the most important contribution made by the application of axiomatics to a subject matter is the resulting precision in conceptualizing about the subject matter. That is, in order to provide a minimal and intuitively comprehensible set of primitive concepts and axioms, the investigator must have a clear and precise understanding of the subject matter. In general, the primitive concepts and axioms for an axiomatic theory for a subject matter are not unique. In fact, alternate axiomatizations of the same subject matter are usually possible. However, the systematic aspects of axiomatics permit, and even facilitate, the comparison of competing axiomatic theories. Furthermore, the formal nature of axiomatics may result in an expansion of the knowledge of the corresponding subject matter. That is, through the derivation of implications of the theoretical axioms, new facts concerning the subject matter may be uncovered that can be verified subsequently. On the other hand, theoretical implications which are not found to be true of the subject matter eventually may result in modifications of the axiomatic theory. In this way the theory becomes more sophisticated and becomes a more satisfactory account of the subject matter.

Various aspects of the axiomatic method have been extensively studied elsewhere and will not be studied extensively herein. For example, methods

are available that can be used to establish the independence of primitive concepts. Other methods can be used to establish the independence of axioms. The notion of deductive proof is well understood and, hence, it is known what it means to deduce implications within a theory. Consistency of axioms and completeness of axioms can be established or discussed through known methods. The process of constructing arithmetic and non-arithmetic models for an axiomatic theory has also been rigorously studied. Topics such as these are beyond the scope of this paper and thus will not be dealt with.

We are interested in uncovering the structure that underlies any given axiomatic theory. In other words we will focus on the commonality that exists for theories that are labelled axiomatic. More precisely, we seek the general kinds of elements an axiomatic theory must contain. We will identify also the relationships which hold between the kinds of elements within an axiomatic theory. We will develop methods by which the totality of these internal relationships may be identified.

Our first task then is to develop a precise description of axiomatic theories. This task is viewed as being one of describing a scientific methodology as it is actually employed. In this sense we are formulating a theory of an empirical phenomena. It is natural then to employ axiomatics in order to provide this description. In fact, we will provide an axiomatic treatment of axiomatic theories. We assume as given as tools in this endeavor both Symbolic Logic and Set Theory. It should be noted that we are not asserting that any axiomatic theory must be formulated in terms of Symbolic Logic and Set Theory; these are simply the tools we choose to

work with. Indeed, it may be that a particular theory can be formulated without making set references obviating the need for Set Theory. Thus, the theory of axiomatic theories developed herein, hopefully, will be applicable to axiomatic theories no matter what language and tools are used within those theories.

In order to discuss the theory of axiomatically structured theories to be developed, let us assume that we have some axiomatic theory, $\mathcal{A}$, available as an example. The theory $\mathcal{A}$ may be an axiomatic development of geometry, Boolean Algebra, Logic, kinship relationships, a psychological measurement scale, or some other subject matter. The theory of axiomatically structured theories is based on eight primitive concepts. This means that our sample theory $\mathcal{A}$ will contain, or will exhibit, each of these eight notions.

First, a set of Primitive Expressions, denoted by PE, is assumed to be available within any axiomatic theory. These expressions correspond to, or are used to represent, the primitive concepts within a given axiomatic theory. That is, for the theory $\mathcal{A}$, PE is the set of expressions that are used to represent the primitive concepts of $\mathcal{A}$. It is important to note that PE is not the set of primitive concepts for $\mathcal{A}$; but rather the syntactical constructions used to represent these concepts.

It may be felt that the term "primitive symbols" is more appropriate than "Primitive Expression". However, the use of the word "symbol" may be taken to imply the use of symbols within the axiomatic theory. For example, the symbol ">" may be used to represent the concept of greater than in some axiomatic theory. An axiomatization could be carried out by

writing "greater than" rather than ">". If this were the case, then the phrase "primitive symbols" would seem to be too restrictive since we would hesitate to call "greater than" a symbol. It is felt that the term "expression" in "Primitive Expression" renders the latter term more general, thereby encompassing under this term the meanings expressed by "primitive symbol", "primitive term", etc. In other words, "Primitive Expression" is preferred since the connotation of "expression" is neutral with respect to stylistic and linguistic conventions.

In addition to the Primitive Expressions, there may be available a set of Secondary Expressions, denoted by SE. The elements of SE correspond to the defined concepts in an axiomatic theory. The elements of SE are not the defined concepts within a theory *u*, but rather the linguistic expressions used to represent these concepts. As in the case of PE, the phrase "Secondary Expression" is preferred due to its generality when compared to phrases such as "secondary symbols", "secondary terms", etc. It is entirely possible that an axiomatic theory, *a*, will not contain, or will not make use of, defined concepts. In this case, SE will be the empty set.

An axiomatic theory will be developed within the context of some particular "language". This language may be a natural language such as English or Russian. The language may also be a formal language utilizing expressions composed from Symbolic Logic and Set Theoretic terms and symbols. This underlying language is intended to serve as the vehicle within which theoretical statements--e.g., axioms and theorems--for the axiomatization are formulated. However, the underlying language must be modified somehow in order to permit references to the theoretical concepts that are used

in the axiomatization. The elements in PE and SE provide these theoretical references. The phrases in PE and SE, and how these phrases are used can be seen to provide the necessary modifications of L.

We can make these notions more precise by employing the theory of transformational grammar. We assume the existence of a language, L, based on a grammar that includes a lexicon. The grammar for L generates the well formed expressions in L from the transformation rules and from the lexicon. The underlying language referred to above is L and as such L contains no theoretical statements for the axiomatization. Theoretical statements for the axiomatization are formulated by modifying the grammar for L as follows. Expressions in either PE or SE not already in the lexicon of L are added to the lexicon of L. Transformation rules for these new lexicon members are also added to the grammar for L.

Expressions in PE or SE already in the lexicon of L need not be added to the lexicon. But these expressions are now intended to be used theoretically as well as in their former non-theoretical role. The theoretical uses of such terms are normally noted through some syntactic device such as capitalization, underscoring, etc. Special transformation rules may or may not be required for these expressions. If the theoretical use is syntactically identical to the non-theoretical use, no alterations of the transformational rules are needed. If the theoretical and non-theoretical uses vary syntactically then special transformation rules must be added to govern the theoretical uses of the expression. In any event the non-theoretical uses of expressions in L will remain unaltered. Thus, every statement expressible in L remains expressible after elements of PE and SE are added. In this sense then an axiomatic theory extends or augments our

underlying language to permit references to theoretical entities and concepts.

Whence, we see that PE and SE are used to generate a new language, L^1 , from L . L^1 , and not L , is the language in which theoretical statements for the subject matter are formulated. These statements are elements of the set of Possible Statements of L^1 . We will denote the set of all Possible Statements within the new language L^1 by PS. Thus, PS represents the set of well formed statements in L^1 , or the set of terminal strings generated by the grammar for L^1 . In general PS will be an infinite set.

When a particular axiomatic theory is developed, a set of Theoretical Statements is in fact formulated. That is, axioms, normally at least, and perhaps definitions and theorems, as well, are actually formulated. These are the Theoretical Statements within the axiomatic theory. This set of statements will be denoted by TS. While TS is a subset of PS, PS and TS must be clearly distinguished. The elements of PS are permissible but perhaps not actually formulated. On the other hand, elements of TS are not only permissible statements but also actually formulated.

The term Theoretical Statement must not be interpreted in such a way to imply that an element of TS must contain either a Primitive or a Secondary Expression. For example, within an axiomatization of kinship relationships it may be desirable to assert some theorem from Set Theory. Such a theorem would be a Theoretical Statement, formulated in L only, even though it contained no Primitive or Secondary Expression.

In addition to the four sets, and the language, L , described above we also will need three relationships. The expression Fxy will be interpreted to mean "expression x is used to formulate statement y ." In the expression Fxy , x will be assumed to be either a Primitive Expression or

a Secondary Expression, and y will be assumed to be a member of the set of Permissible Statements.

The expression $A \vdash x$ will be interpreted to mean "statement x is derived from the set of statements A ". Emphasis should be placed on the "is derived" component of this terminological convention. That is, we will want to be able to refer to the theorems (elements of TS) that are actually formulated and proved within the axiomatization. Given the theorems we will also want to be able to discuss the Theoretical Statements (axioms, definitions and theorems) that are actually used to derive particular theorems. Thus, in the expression $A \vdash x$, A will be a subset of TS and x will be a member of TS.

We will also need to be able to discuss the possibility of deriving possible statements from sets of Possible Statements. For this purpose we use the expression $A \models x$ to mean "statement x can be derived from the set of statements A ". In the expression $A \models x$, A will be a subset of PS and x will be a member of PS.

To summarize, the developments to follow are based on eight primitive concepts:

L , the underlying language

$PE = \{\text{primitive expressions}\}$

$SE = \{\text{secondary expressions}\}$

$PS = \{\text{possible statements}\}$

$TS = \{\text{theoretical statements}\}$

Fxy for "expression x is used to formulate statement y "

$A \vdash x$ for "statement x is derived from the set of statements A "

$A \models x$ for "statement x can be derived from the set of statements A "

As mentioned above we will use both Logic and Set Theory in our theoretical developments. Many theoretical statements herein will be expressed in both verbal form--to aid in their comprehension--and in symbolic form--to ensure their precision. We are assuming an understanding of Set Theory and Logic. Hence, we need only list our notational conventions with a minimum of explanation.

$\forall x$ - universal quantifier - read as "for all x ".

$\exists x$ - existential quantifier - read as "there is at least one x ",
"for some x ", etc.

$\neg$ - negation - read as "not", "it is not the case that", etc.

$\&$ - and - for conjunction

$\vee$ - or - for inclusive disjunction

$\rightarrow$ - if, then - for implication

$\leftrightarrow$ - if and only if - for equivalence

$x \in A$ - set membership - read as " x is a member of set A ".

$x \notin A$ - for not $x \in A$ - read as " x is not a member of A ".

$A \subseteq B$ - subset - means "set A is a subset of set B or set A is equal to set B " and read as " A is a subset of B ".

$A \subset B$ - proper subset - means " A is a subset of B and A is not equal to B " and read as " A is a proper subset of B ".

$A \cup B$ - union - means "the set composed of elements from either A or B " and read as "the union of A and B " or " A union B ".

$A \cap B$ - intersection - means "the set composed of elements from both A and B " reads as "the intersection of A and B " or " A intersect B ".

$A \sim B$ - complement - means "the set composed of elements from A excluding elements from B" read as "A complement B" or "A not B."

$A \times B$ - product - means "the set of ordered pairs constructed by taking elements of A as the first member and elements of B as the second member" and read as "A cross B" or "the product of A and B".

$\{x: Q(x)\}$ - set - means "the set of all elements x which satisfy the statement $Q(x)$ " and read as "the set of all x such that $Q(x)$ ".

$P(A)$ - power set - means "the set consisting of all subsets of the set A" and read as "the power set of A".

$\emptyset$ - empty set - means "the set with no members" and read as "the empty set".

$=$ - equality - read as "equals"

$\neq$ - read as "not equal".

$g: A \rightarrow B$ - means "g is a function from A to B" also read as "g maps A to B."

g is 1-1 - means "function g is a one-to-one function."

g^{-1} - means "the inverse of function g" used only when g is a 1-1 function.

$g(x)$ - means "the image of x under the function g."

As an example of the use of these notational conventions we can formally define the power set as follows.

$$P(A) = \{B: B \subseteq A\}.$$

The variables contained in the three relationships we have identified in axiomatic theories are constrained as follows.

$$F \subseteq (PE \cup SE) \times (PS)$$

$$\vdash \subseteq P(TS) \times (TS)$$

$$\models \subseteq P(PS) \times (PS)$$

We now turn to the axioms that characterize an axiomatically structured theory. After introducing preliminary axioms, definitions and results, we will investigate the syntactic and deductive structure exhibited by axiomatic theories. The notion of Dependency is defined and investigated with a method being described that leads to the identification of all dependencies that exist between axiomatic components. Next, axioms concerning the eliminability of defined concepts and non-creativity of definitions are introduced. These axioms lead to our final developments which culminate in a theorem asserting that all defined concepts can be eliminated from an axiomatic theory.

Axiom 1. PS is the set of well formed statements within some language L^1 , which is a modification of the underlying language L, effected as follows.

- (i) Expressions appearing in PE or SE that do not appear in the lexicon of L are added to the lexicon of L; and transformational rules appropriate for these terms are added to the grammar for L.
- (ii) Transformation rules in the grammar for L for expressions appearing in either PE or SE, and in the lexicon of L,

are modified as required to permit new transformational possibilities as appropriate.

(iii) Step (ii) is carried out in such a way that non-theoretical uses of these expressions remain unchanged so that every statement expressible in L remains expressible in L^1 .

(iv) Primitive and Secondary Expressions as members of the lexicon of L are not in and of themselves Possible Statements.

Axiom 2. The Theoretical Statements are a finite subset of the Possible Statements and the elements of TS are distinctly labeled so that different elements have different labels.

Axiom 3. There is at least one and at most finitely many Primitive Expressions.

Symbolically: PE is finite and $PE \neq \emptyset$.

Axiom 4. The same Expression is not used to represent both a Primitive and Secondary concept.

Symbolically: $PE \cap SE = \emptyset$.

Axiom 5. There is a one-to-one function from the set of Secondary Expressions into the set of Theoretical Statements.

Symbolically: $f: SE \rightarrow TS$ and f is 1-1.

The meaning of axiom 1 should be clear from the discussion preceeding the statement of these five axioms.

Axiom 2 assures us that the Theoretical Statements within an axiomatization are well formed statements within the extended language L^1 . Axiom 2 also asserts that there are at most a finite number of Theoretical Statements in any axiomatization. This is reasonable since only a finite number of axioms, definitions and theorems can be formulated within an axiomatization. It may be felt that this does not allow for elements such as axiom schema which have an infinite number of instances. However, notice that the axiom schema itself counts only as one statement within the axiomatization. Each instance of the axiom that is asserted counts as one more Theoretical Statement. Since only a finite number of instances of such a schema can be asserted, it follows that, even allowing for axiom schema, TS must be a finite set.

Axiom 2 also assures us that individual Theoretical Statements have unique labels. Axiomatic statements are usually labeled as axioms, definitions and theorems with numerical indices identifying each of these elements uniquely. It is entirely possible for the same sentence to appear more than once as a Theoretical Statement. For example, the same sentence may be both a definition and a theorem; or it may be both an axiom and a theorem. To distinguish these occurrences, the Statements must be labeled differently. This portion of Axiom 2 plays a minimal role in the developments to follow; but it is required for the proof of the last, and most important, theorem in this section: eliminability of defined concepts.

Axiom 3 limits the set of Primitive Expressions within a theory to a finite number of elements. Axiom 3 does not eliminate the possibility of taking an infinite set as a primitive concept. An infinite set, e.g., the natural numbers, would be referred to by one expression, say N. Axiom 3 also asserts that the set of Primitive Expressions must be non-empty. If $PE = \emptyset$, then there is no conceptual basis for the axiomatization. Thus, it must be the case that $PE \neq \emptyset$.

As asserted in Axiom 4 the set of Primitive Expressions must be disjoint from the set of Secondary Expressions. This is reasonable since the primitive concepts within a theory play a function that is clearly distinct from the function played by defined concepts. Primitive concepts, represented by PE, are intended to be intuitively comprehensible. Defined concepts, represented by SE, are syntactical abbreviations that simplify the axiomatization. Elements of SE, as to be discussed in detail, are entirely expendable since they always can be replaced with expressions built up from PE alone.

It is expected that every defined expression will be used in some way within an axiomatization. Otherwise, the theory is unnecessarily obfuscated through the inclusion of unemployed defined conceptualization. Defined concepts require a definition which will introduce the associated Secondary Expression. Thus, in general, every Secondary Expression will be used to formulate at least one Theoretical Statement, its Definition. This is the rationale that underlies Axiom 5 above, our first definition and theorems which follow, and Axiom 6, which also follows. Axiom 5 serves to associate every Secondary Expression with a unique Theoretical Expression to be called its definition.

As will be seen, Axiom 5 eliminates the possibility of defining more than one Secondary Expression in one Theoretical Statement. In order to discuss the eliminability of defined concepts and non-creativity of definitional statements, we need to be able to refer to the definition for each Secondary Expression. Such a definition will be a statement within the axiomatization. If we allow more than one Secondary Expression to be defined in one Statement, then we require some method of separating and distinguishing the definitions. Such a method does not seem to be available. For example, it is not possible to separate a multiple definition by syntactic devices such as separating sentences. Recursive definitions habitually have more than one syntactic component and, in fact, are usually expressed through two sentences: the conglomerate appropriately being labeled as one Statement. Basing the characterization of definitions on the one-to-one function postulated in Axiom 5 seems to be the most reasonable approach available.

It is recognized that in practice this approach is often violated; but in an insignificant way. An axiomatization may have a labeled definition in which two or more concepts are defined. For example, in his axiomatic treatment of Set Theory (1960) Suppes defines $n-1$ and $n-2$ for ordinal numbers in one definition (his Definition 12, p. 141). In this case, however, it is readily seen that the definition can be split into two distinct definitions, preserving the validity of our approach. In general, it will be true that definitions introducing multiple concepts can be replaced by a series of definitions with each definition introducing only one Secondary Expression. Thus, while artistic preference may lead to violation of this approach, we contend that these violations are trivial

in nature, and that, in fact, it is reasonable to base the concept of a definition on Axiom 5. In other words, we might say "the letter, but not the spirit of the approach we take is sometimes violated in practice".

Definition 1. The image of a Secondary Expression, x , under the one-to-one function f , postulated in Axiom 5, is referred to as the Definition of x and is denoted by $\bar{x}$.

Throughout the remainder of this paper we will refer to the function postulated in Axiom 5 as " f ." Thus, from Definition 1 we know that for all x in SE, $f(x) = \bar{x}$. Since f is a 1-1 function for all x, y in SE if $x \neq y$, then $f(x) \neq f(y)$; and, thus, $\bar{x} \neq \bar{y}$. We state this as a theorem.

Theorem 1. Distinct Secondary Expressions have distinct definitions.

Symbolically: $(\forall x, y \in SE)(x \neq y \rightarrow \bar{x} \neq \bar{y})$.

From Axiom 2 we know that TS is finite. Since F is one-to-one from SE into TS it follows that SE is also finite.

Theorem 2. There are finitely many Secondary Expressions; or SE is finite.

Since $\bar{x}$ is the definition of Secondary Expression x for all x in SE, we expect x to appear in $\bar{x}$. That is, we expect x to be used to formulate its own definition. This is the substance of Axiom 6.

Axiom 6. Every Secondary Expression is used to formulate its own Definition.

Symbolically: $(\forall x \in SE) Fx f(x)$.

Axiom 6 obviously can be written as:

$(\forall x \in SE) Fx \bar{x}$.

Notice that we do not require every Primitive Expression to be used in a statement. That is, it may be that some Primitive Expressions are never utilized. While this would be an abnormal approach to axiomatics, it would not violate any rigid methodological principles. Notice also that these notions concerning Definitions are restricted to elements of SE only, excluding elements of PE. Even when a Primitive Expression is used to formulate several Statements none of these elements are to be construed as the definition of the expression. A primitive concept may be used in several axioms and it is incorrect to select any one if these as the definition of the concept. The entire collection of axioms in a sense characterize the primitive concepts and their interrelationships; but they do not define these notions. The Primitive Expressions in an axiomatization represent the undefined, intuitively comprehensible conceptual basis for the theory.

Our next assumption concerning Definitions is simply that they are unproved statements. Following Suppes, we perceive definitions as functioning like axioms within an axiomatic theory. Definitions serve as additional suppositions for subsequent developments. Whence, it is entirely reasonable to assume that Definitions are not derived.

Axiom 7. Definitions are not derived.

Symbolically: $(\forall x \in SE)(\forall A \in \mathcal{P}(TS)) \neg (A \vdash f(x))$.

Axiom 7 can be written as follows:

$$(\forall x \in SE) \neg (\exists A \in \mathcal{P}(TS)) (A \vdash x).$$

We now turn to an investigation of the structural properties of axiomatic theories. Two more axioms will be introduced in the course of these developments. The last three axioms will be introduced later in the paper when an investigation of eliminability is undertaken.

Axiomatic theories exhibit two kinds of structure: syntactic and deductive. We begin with a discussion of syntactic structure. Syntactic structure results from sequences of the "used in the formulation of relationship." Within the phrase Fxy , x refers to Expressions, Primitive or Secondary, and y refers to Statements. Thus, for a sequence of the form

$$Fwx, Fzy$$

to arise connecting x to y , it must be the case that w is the Definition of z . That is, the only way to link two elements-- x an Expression and y a Statement--through a series of syntactic relationships based on F , is to have the series composed from Definitions and their defined Expressions. In other words, we will focus on series of the following form

$$Fx\bar{z}, Fzy$$

or more generally,

$$Fx\bar{x}_1, Fx_1\bar{x}_2, \dots, Fx_{n-1}\bar{x}_n, Fx_n y.$$

It is natural in such a case to say that y is Syntactically Dependent on x . Formally, we have the following definition.

Definition 2. A Theoretical Statement, s , is Syntactically Dependent on x , either a Primitive or Secondary Expression, if and only if

(i) s is not the Definition of x : $s \neq \bar{x}$

(ii) there is a sequence, possibly empty, of Secondary Expressions

$x_1, \dots, x_n$ such that

$$Fxx_1 \ \& \ Fx_1x_2 \ \& \ \dots \ \& \ Fx_{n-1}x_n \ \& \ Fx_ns$$

holds.

Definition 2 applies to any Primitive Expression, any Secondary Expression and any Theoretical Statement. Thus, if x is a Primitive Expression and s is any Theoretical Statement such that Fxs holds, then s is Syntactically Dependent on x . Also, if x is any Secondary Expression and s is any Theoretical Statement such that $s \neq \bar{x}$, and Fxs holds, then s is Syntactically Dependent on x . Definition 2 implies that $\bar{x}$ is not Syntactically Dependent on x for x in SE. The definition of Syntactic Dependency has been formulated in this way in order to simplify the formulation of some of the results that follow. Part (i) in Definition 2 could be deleted eliminating this result. However, as a result some of the following results would be unnecessarily complicated, and, thus, we will retain condition (i). We have the following theorem.

Theorem 3. The Definition of a Secondary Expression is not Syntactically Dependent on the Expression defined in the Definition.

We expect the "used in the formulation of" relationship to be non-cyclic. That is, it would be incorrect to use one defined concept to define a second; the second being used to define a third; the third a fourth; and so on to an n^{th} concept used to define the first. Definitional circularity of this sort is eliminated in our next axiom.

Axiom 8. The "used in the formulation of" relationship is non-cyclic in that there are no sequences of the following form where $n \geq 1$, and $x, x_1, \dots, x_n$ are all members of SE.

$$Fxx_1 \bar{x} \ \& \ Fx_1x_2 \bar{x}_2 \ \& \ \dots \ \& \ Fx_{n-1}x_n \bar{x}_n \ \& \ Fx_n \bar{x}.$$

In Axiom 8 notice that we do not need to consider the possibility of x in PE since such an x does not have a Definition. Nor do we need to consider the case where $\bar{x}$ is a Statement other than a Definition: Statements cannot be used to formulate other Statements.

Theorem 4. Syntactic Dependency exhibits a form of transitivity: For all x in SE, for all y in $PE \cup SE$ and for all s in TS; if s is Syntactically Dependent on x and if $\bar{x}$ is Syntactically Dependent on y , then s is Syntactically Dependent on y .

To establish Theorem 4 assume

s is Syntactically Dependent on x

and

$\bar{x}$ is Syntactically Dependent on y .

This means that we have both

$$Fxx_1 \bar{x} \ \& \ \dots \ \& \ Fx_n \bar{x}_n$$

and

$$Fyy_1 \bar{y} \ \& \ \dots \ \& \ Fy_m \bar{y}_m.$$

We obviously have the required sequence from y to s rendering s Syntactically Dependent on y .

Notice in the above sequences that $s \neq \bar{x}$ and $y \neq x$. Since these conditions hold we are assured that the conclusion of Theorem 4 does not imply contradictorily that $\bar{x}$ is Syntactically Dependent on x . We also know that $s \neq \bar{y}$. If $s = \bar{y}$ we have both

$$Fxx_1 \bar{x} \text{ \& } \dots \text{ \& } Fx_n \bar{y}$$

and

$$Fyy_1 \bar{x} \text{ \& } \dots \text{ \& } Fy_m \bar{x}$$

where $n \geq 1$ and $m \geq 1$. This result contradicts Axiom 8. Thus again we are assured that the conclusion of Theorem 4 does not contradict Theorem 3.

We expect Syntactic Dependency to be non-cyclical in that we do not want an element to Syntactically Depend on itself through a chain of Syntactic Dependency relationships. Cyclic Syntactic Dependency would be a chain of the following form.

$$\begin{array}{l} \bar{x}_1 \text{ is Syntactically Dependent on } x \\ \bar{x}_2 \text{ is Syntactically Dependent on } x_1 \\ \vdots \\ \bar{x}_n \text{ is Syntactically Dependent on } x_{n-1} \\ \bar{x} \text{ is Syntactically Dependent on } x_n \end{array}$$

Applying Theorem 4 repeatedly in this case implies that $\bar{x}$ is Syntactically Dependent on x contradicting Theorem 3. The simple case of a one-step cycle chain is ruled out by the definition of Syntactic Dependency. This establishes the following theorem.

Theorem 5. Syntactic Dependency is non-cyclic.

Our next definition introduces the notion of Deductive Dependence which is based on the relationship " $\vdash$ ". Suppose we have two Theoretical Statements s, s_1 and a set A such that

$$s_1 \in A, A \vdash s$$

both hold. In such a case it would be natural to say that s is Deductively Dependent on s_1 . The more general case arises when we have the following conditions.

$$\begin{array}{ll} s_1 \in A_1 & A_1 \vdash s_2 \\ s_2 \in A_2 & A_2 \vdash s_3 \\ & \vdots \\ s_{n-1} \in A_{n-1} & A_{n-1} \vdash s_n \\ s_n \in A_n & A_n \vdash s \end{array}$$

Again, we would assert in this case that s is Deductively Dependent on s_1 since we have a sequence of "is derived from" relationships leading from s_1 to s .

Definition 3. A Theoretical Statement, s , is Deductively Dependent on a Theoretical Statement s_1 if and only if there is a sequence (possibly empty) of Theoretical Statements $y_1, \dots, y_{n-1}$ and a sequence of sets $A_1, \dots, A_n$ ($n \geq 1$) such that the following relationships hold:

$$A_1 \vdash y_1, A_2 \vdash y_2, \dots, A_{n-1} \vdash y_{n-1}, A_n \vdash s$$

where

$$s_1 \in A_1, y_1 \in A_2, \dots, y_{n-2} \in A_{n-1}, y_{n-1} \in A_n.$$

We expect Deductive Dependency to be non-cyclic. That is, it would be incorrect to have one statement included in a set used to derive a second statement; the second in a set used to derive a third; and so on up to an n^{th} statement in a set used to derive the first. This classical form of circular reasoning must be avoided. In the simplest case we want to eliminate the following possibility.

$$\begin{array}{ll} s \in A_1 & A_1 \vdash s_1 \\ s_1 \in A_2 & A_2 \vdash s \end{array}$$

In this case we have both

s_1 is Deductively Dependent on s

and

s is Deductively Dependent on s_1 .

Together these statements indicate a symmetry for Deductive Dependency. Since symmetry of this type is to be avoided, we rule it out with an axiom. In effect the next axiom rules out cyclic chains of the "used to derive" relationship.

Axiom 9. Deductive Dependency is an asymmetric relationship.

From Axiom 9 we can prove that Deductive Dependency is irreflexive: Theorem 6 below. That is, s is not Deductively Dependent on itself.

Deductive Dependency is also transitive as stated in the next theorem. This fact follows directly from the definition of Deductive Dependency.

Theorem 6. Deductive Dependency is an irreflexive, transitive relationship.

We can now show that Deductive Dependency is non-cyclic. A cyclic chain of Deductive Dependencies would result in the following conditions.

$$\begin{array}{l} s_1 \text{ is Deductively Dependent on } s \\ s_2 \text{ is Deductively Dependent on } s_1 \\ \vdots \\ s_n \text{ is Deductively Dependent on } s_{n-1} \\ s \text{ is Deductively Dependent on } s_n \end{array}$$

From Theorem 6 (transitivity) these conditions imply that s is Deductively Dependent on s , contradicting the fact that Deductive Dependency is irreflexive (Theorem 6). Thus, we have the following theorem.

Theorem 7. Deductive Dependency is non-cyclic.

The notions of Syntactic and Deductive Dependency can be combined to form the concept of Dependency. If s is Syntactically Dependent on x or if s is Deductively Dependent on x , then s is Dependent on x . In the first of these cases, $s \in TS$ and $x \in PEUSE$. In the second case $s \in TS$ and $x \in TS$. Notice also that it may be the case that s is Deductively Dependent on y and y is Syntactically Dependent on x . In this eventuality, we would also assert that s is Dependent on x , where $s \in TS$, $y \in TS$ and $x \in PEUSE$. The case

where s is Syntactically Dependent on y and y is Deductively Dependent on x cannot arise since y cannot be both a Primary or Secondary Expression, and a Theoretical Statement--Axiom 1, Part (iv). The formal definition of Dependency is as follows.

Definition 4. A Theoretical Statement s is Dependent on x , if and only if one of the following conditions hold.

- (i) s is Syntactically Dependent on x , and $x \in \text{PEUSE}$.
- (ii) s is Deductively Dependent on x , and $x \in \text{TS}$.
- (iii) There is a Theoretical Statement s_1 such that s is Deductively Dependent on s_1 and s_1 is Syntactically Dependent on x where $x \in \text{PEUSE}$.

Dependency will be asymmetric, transitive and non-cyclic as stated in the next three theorems.

Theorem 8. Dependency is an asymmetric relationship.

Theorem 9. Dependency is a transitive relationship.

Theorem 10. Dependency is a non-cyclic relationship.

To prove Theorem 8, let s be Dependent on x , which means $s \in \text{TS}$, and either $x \in \text{TS}$ or $x \in \text{PEUSE}$. If $x \in \text{TS}$ we know that s must be Deductively Dependent on x . By Axiom 9, x is not Deductively Dependent on s . Also no element (including x) can be Syntactically Dependent on s , so x cannot be Dependent on s . Finally, if $x \in \text{PEUSE}$ then x cannot be Dependent on any elements, including s . Therefore, if s is Dependent on x , then x is not Dependent on s , establishing Theorem 8.

For Theorem 9, suppose s_2 is Dependent on s_1 and s_1 is Dependent on s . Thus, $s_2 \in TS$ and $s_1 \in TS$ which implies s_2 is Deductively Dependent on s_1 . If s_1 is Deductively Dependent on s , then s_2 is Deductively Dependent on s by virtue of Theorem 6. Whence, s_2 is Dependent on s . If s_1 is Syntactically Dependent on s , then s_2 is Dependent on s by condition (iii) of the definition of Dependency. The last possibility is that s_1 is Deductively Dependent on some element s_3 and s_3 is Syntactically Dependent on s . In this case, s_2 is Deductively Dependent on s_3 and, applying condition (iii) of the definition of Dependency, yields s_2 is Dependent on s .

To prove Theorem 10 assume the following conditions hold.

$$\begin{array}{l} s \text{ is Dependent on } s_1 \\ s_1 \text{ is Dependent on } s_2 \\ \vdots \\ s_{n-2} \text{ is Dependent on } s_n \\ s_n \text{ is Dependent on } s \end{array}$$

With these conditions, Theorem 9 implies that s is Dependent on s . But Theorem 8 entails the irreflexivity of Dependency, eliminating this possibility.

While Definition 4 provides us with the appropriate characterization of Dependency, Definition 4 does not provide much help in identifying all Dependency relationships within an Axiomatically Structured Theory. Having a complete description of the Dependency relationship is desirable for many reasons. Consider, for example, the situation where one person has available an axiomatic theory of a particular subject matter and another

person desires to learn that theory. In what order can the Theoretical Statements be explained to the learner? It seems reasonable to permit the introduction of any Theoretical Statement as long as every Term on which the Statement is Syntactically Dependent has been defined; and as long as all Theoretical Statements on which the Statement is Deductively Dependent have been introduced. The details of the discussion of this problem are beyond the scope of this paper. Rather than pursue these notions, we turn to the development of a procedure for identifying all Dependency relationships.

A matrix method based on graph theoretical principles has been developed by Harary et al (1965) that can be adopted to provide a mechanical procedure for identifying all Dependency relationships. Briefly, this method assumes the existence of n elements that are interrelated by 1 or more non-cyclic relationships. An adjacency matrix is developed for each relationship indicating by a row and column entry that the row element is related directly to the column element. A matrix entry of 1 serves to indicate the existence of this direct (adjacent) relationship. The matrices are then added element by element. The resultant matrix is then reduced to 0's and 1's by setting an entry to 1 just in case the sum for that row and column is greater than or equal to 1. Thus, the resultant matrix has a row, column entry equal to 1 just in case at least one of the original relationships relates the row element to the column element. Notice that from the reduced resultant matrix we can neither tell what relationship relates the elements, nor can we tell how many relationships relate the elements.

When the matrices are based on the Syntactic and Deductive Dependency relationships, the reduced resultant matrix shows all adjacent Dependency relationships that exist between the n elements. This matrix is an $n \times n$

matrix and can be squared. The square of the matrix indicates for any two elements the number of 2-step, or 2-link, Dependency chains that connect the elements. This matrix can be reduced to 0's and 1's by the procedure used before. In general, the m^{th} power of the original reduced matrix shows for each two elements the number of m-step, or m-link, Dependency chains that connect the elements.

For some m it can be shown that the $m+1^{\text{st}}$ power of the original reduced matrix must be the zero matrix. If we add all the powers of the original reduced Dependency matrix up to and including m , the resulting matrix indicates all Dependency relationships. Suppose we reduce this final matrix to 0's and 1's as before. We can show that in this matrix a row, column entry is 1 just in case the row element is related to the column element by a sequence of the original relationships. The matrix does not tell us what relationships make up this sequence; it does not tell us how many relationships are in the chain; and it does not tell us the order in which the relationships appear in the chain. All the matrix tells us is that there is some chain of relationships from the row element to the column element, i.e. the column element is Dependent on the row element.

We now formalize the procedure outlined above. We begin with the formal definition of an adjacency matrix.

Definition 5. The adjacency matrix for an arbitrary relationship R , defined between n elements, is the $n \times n$ square matrix $A(R)=[a_{ij}]$ where $a_{ij}=1$ if element i is related to element j by R , and $a_{ij}=0$, otherwise.

We have two relationships on which Dependency is based: "used to formulate" and "is derived from". We will use these relationships to develop two $n \times n$ adjacency matrices to be used in the procedure sketched above. First consider the "used to formulate" relationship. Consider the set made up of elements from PE and from TS (i.e., PEUTS); and let n be the number of elements in this set. Order these n elements in some way. As to be discussed below, we do not distinguish Secondary Expressions from their Definitions in this list.

Definition 6. The Adjacency Matrix for Syntactic Dependency is defined as the $n \times n$ matrix $A(SD)=[a_{ij}]$ which is constructed by the following three rules.

- (1) If element i is in PE, and element j is in TS, then $a_{ij}=1$ if element i is used to formulate element j .
- (2) If element i is a Definition and element j is in TS, and if $i \neq j$, then $a_{ij}=1$ if the Secondary Expression defined in element i is used to formulate element j .
- (3) All other entries are zero.

Of the three rules in Definition 6 only rule (2) requires some explanation. As mentioned above we do not distinguish Secondary Expressions and their Definitions in the Adjacency Matrix for Syntactic Dependency. That is, we have simply listed elements from PE and TS, but we have not included elements from SE since the Definitions are already listed. Thus, when we consider a row element that is a Definition we do not ask if that element is used to formulate a member of TS. Definitions are Statements and, as such, are not used to formulate other Statements. What we must ask is whether or not the Secondary Expression being Defined is used to formulate other Statements, as we have done in rule (2).

A simple example will show that this procedure is the correct procedure. Consider three Definitions of three Secondary Expressions where the first is used to formulate the Definition of the second and the second is used to formulate the Definition of the third. Definition 6 leads in this case to the following matrix (ignoring Primitive Expressions and other possible Theoretical Statements).

	D1	D2	D3
D1	0	1	0
D2	0	0	1
D3	0	0	0

Squaring this matrix yields

	D1	D2	D3
D1	0	0	1
D2	0	0	0
D3	0	0	0

which shows (by the above outlined procedure) that the third Definition is Syntactically Dependent on the first Secondary Expression as expected.

Suppose contrary to Definition 6 we distinguish between Secondary Expressions and Definitions. This would lead to the following adjacency matrix showing all instances of the "used in the formulation of" relationships.

	SE1	SE2	SE3	D1	D2	D3
SE1	0	0	0	1	1	0
SE2	0	0	0	0	1	1
SE3	0	0	0	0	0	1
D1	0	0	0	0	0	0
D2	0	0	0	0	0	0
D3	0	0	0	0	0	0

The square of this matrix is the zero matrix or the matrix containing only zeroes and does not show that Definition 3 is Syntactically Dependent on the first Secondary Expression. Whence, this approach is in error and, appropriately, Definition 6 avoids this pitfall.

By rule (2), and in the above example, we do not indicate in the Syntactic Dependency Adjacency Matrix that a Secondary Expression is used to formulate its own Definition. By definition we know that a Definition is Syntactically Dependent on the Secondary Expression it defines. Thus, there are no ones along the diagonal of the matrix $A(SD)$. In any event, if our interest in Dependency is motivated by a desire, for example, to insure that all elements an element depends upon are explained first, then to discover that a Definition depends upon the Secondary Expression it defines is of little value. The Definition is intended, after all, to introduce the Secondary Expression into the theory. Thus, knowing that this particular Dependency exists will not result in the prior introduction of any Theoretical Statements. This provides further justification for the exclusion of $\bar{x}$ being Syntactically Dependent on x in the definition of Syntactic Dependency. This is in contrast to a case in which a Definition

of a Secondary Expression Depends on other Secondary Expressions. In this case, we would make sure the other expressions are Defined first before proceeding to the new Definition.

From a procedural point of view we must not indicate that a Secondary Expression is used to formulate its own Definition. To do so means that a 1 will be entered along the diagonal of the adjacency matrix. This means that the relationship being represented is cyclic. In other words raising the matrix to successive powers never results in the zero matrix and, thus, the procedure to be used will not be applicable to the problem at hand. Whence, Definition 6 results in an adjacency matrix for a non-cyclic relationship (Theorem 5)--as required.

The adjacency matrix for Deductive Dependency is defined in terms of the same n elements, arranged in the same order, as used to define the Adjacency Matrix for Syntactic Dependency.

Definition 7. The Adjacency Matrix for Deductive Dependency is defined as the $n \times n$ matrix $A(DD)=[b_{ij}]$ which is constructed by the following rules.

- (1) If element j in PEUTS is derived from a set of statements that includes element i in PEUTS then $b_{ij} = 1$.
- (2) All other entries are zero.

The meaning of Definition 7 is self-evident. Notice, however, that the rows and columns in $A(DD)$ correspond to the rows and columns in $A(SD)$. Also notice that there are no diagonal entries in $A(SD)$ since Deductive Dependency is irreflexive.

For convenience we introduce the notion of Boolean Arithmetic. The only difference between Boolean Arithmetic and non-Boolean Arithmetic is that in Boolean Arithmetic $1+1=1$. Following Harary, et al (1965) we denote Boolean Arithmetic by writing $(n+m)\#$. Thus, $(1+1)\#=2\#=1$ and, in general, it follows that if $n \geq 1$, then $n\#=1$.

Definition 8. For all positive integers n, m $(n+m)\#$ is the Boolean Sum of n and m where $(n+m)\#=0$ if $n+m=0$ and $(n+m)\#=1$ if $n+m \geq 1$.

Definition 9. The Adjacency Matrix for the Dependency relationship, $A(D)$, is defined as

$$A(D) = [A(SD) + A(DD)]\#.$$

To form the Adjacency Matrix for the Dependency relationship the Adjacency Matrices for Syntactic and Deductive Dependency are constructed first. These Adjacency Matrices are then Boolean added element by element. This is normal matrix addition except that the individual sums are Boolean sums so that each entry in $A(D)$ is either a 1 or 0. It follows, that each element, c_{ij} , of $A(D)$ is such that

$$c_{ij} = 1 \text{ if the } j^{\text{th}} \text{ element in PE} \cup \text{TS is either directly Syntactically Dependent or directly Deductively Dependent on the } i^{\text{th}} \text{ element in PE} \cup \text{TS.}$$

and

$$c_{ij} = 0 \text{ if the } j^{\text{th}} \text{ element in PE} \cup \text{TS is neither directly Syntactically Dependent nor directly Deductively Dependent on the } i^{\text{th}} \text{ element in PE} \cup \text{TS.}$$

Whence, $A(D)$ depicts the Dependency relationship to the extent of showing all dependency chains of length 1. The matrix $A(D)$ does not necessarily depict all the Dependency relationships since Dependency may be established through relational chains of length greater than 1. The next theorem provides the rationale needed to generate the complete dependency matrix.

Theorem 11. (Harary, et al, 1965). In $A(D)^\ell$ ($A(D)$ raised to the ℓ^{th} power) the i, j^{th} entry is the number of chains of Dependency relationships of length ℓ leading from the i^{th} element in $PE \cup TS$ to the j^{th} element in $PE \cup TS$.

It can be shown that for some $m > 1$, $A(D)^{m+1} = [0]$, the matrix containing only zeros. This follows since there are no loops in the Dependency relation. Thus, since the number of elements in $PE \cup TS$ is finite, there is an upper limit on the length of possible Dependency relationship sequences leading from one element of $PE \cup TS$ to another. We call this maximum m the Maximal Length Dependency Chain.

At this point in our development we have not defined formally the notion of an Axiomatically Structured Theory. An Axiomatically Structured Theory will be an eight-tuple

$$\mathcal{U} = \langle L, PE, SE, PS, TS, F, \vdash, \models \rangle$$

which satisfies Axioms 1-9 above and Axioms 10-12 below. We assume in the next few definitions and theorems that this notion has been defined. This assumption is perfectly harmless and simply makes the subsequent developments more precise.

Definition 10. The Maximal Length Dependency Chain for an Axiomatically Structured Theory is defined as the minimal m such that $A(D)^{m+1} = [0]$.

Definition 11. The Dependency Matrix for an Axiomatically Structured Theory, $\mathcal{A}$, is defined as

$$D_{\mathcal{A}} = [A(D)^1 + A(D)^2 + \dots + A(D)^m] \#$$

where m is the Maximal Length Dependency Chain for $\mathcal{A}$.

When there is no confusion about which Axiomatically Structured Theory, $\mathcal{A}$, is being discussed we will write D in lieu of $D_{\mathcal{A}}$.

Suppose D is the Dependency Matrix for an Axiomatically Structured Theory and let x, y be elements in $PE\mathcal{U}TS$. Within D , x will correspond to some row and y will correspond to some column. We will write a_{xy} to indicate the entry in D consisting of the row and column corresponding to x and y , respectively. If x is represented by the i^{th} row and y is represented by the j^{th} column in D , then $a_{xy} = a_{ij}$.

Definition 12. a_{xy} denotes the entry in D consisting of the row and column corresponding to x and y respectively for x, y in $PE\mathcal{U}TS$.

Theorem 12. Let x, y be elements in $PE\mathcal{U}TS$. y is Dependent on x if and only if $a_{xy} = 1$.

If y is Dependent on x then there will be a sequence of Dependency relationships leading from x to y . Such a sequence will be composed of say ℓ steps or links where $\ell \leq m$. Thus, the entry representing the x to y Dependency relationship in the ℓ^{th} power of $A(D)$ will be 1. When the Dependency Matrix is formed by Boolean adding the powers of $A(D)$ up to $m \geq \ell$ this entry will carry over making $a_{xy} = 1$. Conversely, if $a_{xy} = 1$, then there will be some power of $A(D)$, say ℓ , where $\ell \leq m$ and the row, column entry corresponding to x and y will be non-zero. Thus there must be at least one Dependency sequence of length ℓ from x to y . That is, y is Dependent on x .

Theorem 13. Let $x \in SE$ and let $y \in TS$. y is Dependent on x if and only if

$$a_{\bar{x}y} = 1.$$

To prove Theorem 13 let $x \in SE$ any $y \in TS$. Suppose first that y is Dependent on x . Rule (2) in Definition 6 assures us that the row entry in the Syntactic Dependency Adjacency Matrix corresponding to $\bar{x}$ will contain a 1 in some column that subsequently links $\bar{x}$ to y via a sequence of Dependency links. This sequence will eventually result in a 1 being entered in the row, column corresponding to $\bar{x}, y$ for some power of the Adjacency Matrix for Dependency. When $A(D)$ is formed this entry will carry over causing $a_{\bar{x}y}$ to be set to 1.

Conversely suppose $a_{\bar{x}y} = 1$. This entry is caused by a 1 entered in some power of the Adjacency Matrix for Dependency. Whence, there is an ℓ step chain of Dependencies from $\bar{x}$ to y where $\ell \leq m$. In other words, y is Dependent on x . Rule (2) in Definition 6 again is used to ensure

us that $\bar{x}$ is linked to some element in the Adjacency Matrix for Syntactic Dependency; and this other element, if not y itself, provides the Dependency links to y .

Theorems 12 and 13 and the concepts on which they are based, provide a direct method for developing the complete Dependency relationship for an Axiomatically Structured Theory.

For the remainder of this paper we will direct our attention to the questions of the eliminability of defined concepts and the non-creativity of definitions. The concepts underlying these notions are quite simple. Consider the question of the elimination of defined concepts. As mentioned earlier, defined concepts are intended to serve as syntactic abbreviations for other, usually more complex, syntactic formulations. For example, in set theory the notion of proper subset is often defined in terms of set equality and the subset relationship:

$$A \subset B \leftrightarrow A \subseteq B \ \& \ A \neq B$$

where A and B are sets. In this statement " $\subset$ " is the Defined Secondary Expression representing "proper subset." To say that " $\subset$ " is eliminable means, first that every occurrence of " $\subset$ " in every Possible Statement within the axiomatic theory can be replaced with a syntactic construction not containing " $\subset$ ". The above "definition" of " $\subset$ " is an equivalence and, thus, every Statement containing " $\subset$ " can be rewritten without " $\subset$ " simply by substituting the expression " $A \subseteq B \ \& \ A \neq B$ " for " $A \subset B$ ". In practice, definitions usually take the form of logical equivalencies or logical identities rendering this substitution process straightforward.

Suppes (1957) provides a formulation of a "Criterion of Eliminability." Suppes' criterion makes use of the phrase: "preceding definitions of the theory." Suppes' perceives the Theoretical Statements in an axiomatic theory as being ordered by this notion of precedence which makes his criteria sensible as long as such an ordering is available. We, however, have not ordered the Theoretical Statements in any way. That is, the only structure we have available results from the notions of Syntactic and Deductive Dependence. These notions are not suitable for the construction of a precedence ordering of the type Suppes requires.

The divergence between Suppes' and our approach can be explained in another way as well. Suppes seems to have based his criteria on some notion of the order in which Theoretical Statements are written down--say in a book--or communicated. Of course, this order may not be the temporal order in which the Statements are formulated. The order is simply the order in which they may be encountered by someone learning the theory. We have chosen, in contrast, to treat the questions of inherent structure and communication separately. In this paper we are treating structure which is not dependent on presentation order. In a subsequent paper we will be concerned with the question of communication of axiomatic theories. As will be seen, the order of communication, or presentation, of Theoretical Statements will depend on the structure exhibited by the theory.

The second component of eliminability is that when a Secondary Expression, x , is eliminated from a Statement, s , resulting in a Statement, s_1 , s and s_1 must be provably equivalent. The equivalence proof must follow from the Definition of the eliminated Expression only. That is,

it must be possible to prove the equivalence of s and s_1 from $\bar{x}$. We do not expect that such a proof has actually been carried out ($\vdash$) for all Secondary Expressions and all Possible Statements. We only require that it can be carried out ($\bar{F}$). Thus, this component of the axiom of eliminability must be written as follows.

$$(\bar{x}) \bar{F}s \leftrightarrow s_1.$$

The analog to this assumption will be found in Suppes' criteria.

Notice also that s_1 need only differ from s in that x in s has been replaced with the Expressions in $\bar{x}$ that defined x . This means that an Expression, w , occurs in s_1 just in case w is not x , and either w is in s or w is in $\bar{x}$. Whence, the syntactic component of the axiom of eliminability is rendered as

$$Fws_1 \leftrightarrow w \neq x \ \& \ (Fws \vee Fw\bar{x}).$$

We have explained the major features of Axiom 11 below--the axiom of eliminability. Before formally stating Axiom 11, we introduce another axiom that simply relates the "is derived" and "can be derived" relationships. While simple and straightforward, this axiom is required for subsequent developments.

Axiom 10. If a Theoretical Statement is derived, then it can be derived.

Symbolically: $(\forall s \in TS)(\forall A \in P(TS))(A \vdash s \rightarrow A \bar{F}s)$

Axiom 11. (Eliminability) The Definition of any Secondary Expression, x , is formulated in such a way that any Possible Statement, s ,

formulated in terms of x is provable equivalent to another Possible Statement, s_1 , that is formulated only in terms of Expressions other than x , and that appear either in s or in $\bar{x}$; the equivalence proof using only $\bar{x}$.

Symbolically: $(\forall x \in SE)(\forall s \in PS)[Fxs \rightarrow (\exists s_1 \in PS)(\forall w \in PEUSE)$

$$[(Fws_1 \leftrightarrow w \neq x \ \& \ (Fws \vee Fw\bar{x}))$$

$$\& ((\bar{x}) \models s \leftrightarrow s_1)]]$$

Before introducing the axiom of non-creativity we introduce a definition and a theorem that establishes a simple result concerning the Expressions used to formulate s and s_1 , where s_1 arises from s by Axiom 11. The theorem asserts that the Expressions used to formulate s_1 will consist of those Expressions occurring either in s or in $\bar{x}$ excluding x . The preliminary definition introduces a theoretical notion that allows us to refer to the Expressions used to formulate any Possible Statement.

Definition 13. For all s in PS,

$$F''(s) = \{x \in PEUSE : Fxs\}$$

Theorem 14. For all s in PS if s_1 is the Possible Statement postulated to exist in Axiom 11, then

$$F''(s_1) = [F''(s) \cup F''(\bar{x})] \sim \{x\}.$$

To prove this theorem, let s, s_1 satisfy the assumptions of the theorem. From Axiom 11 we know

$$Fws_1 \leftrightarrow w/x \ \& \ (Fws \vee Fw\bar{x})$$

for any $w \in PEUSE$. But, we also know

$$Fws_1 \leftrightarrow w \in F''(s_1),$$

$$Fws \leftrightarrow w \in F''(s),$$

and

$$Fw\bar{x} \leftrightarrow w \in F''(\bar{x}).$$

Whence,

$$w \in F''(s_1) \leftrightarrow [w \in F''(s) \vee w \in F''(\bar{x})] \ \& \ w \notin (x)$$

and the proof is complete.

Now consider the notion of non-creativity. Our formulation of non-creativity will be very similar to Suppes' formulation. However, as with eliminability, Suppes employs the precedence concept, unavailable to us, in non-creativity. The idea behind non-creativity is that Definitions should not substantially enhance the deductive power of the axiomatic theory. We must not be able to prove statements concerning the underlying subject matter with a definition that cannot be proved without the definition. That is, definitions being essentially syntactic abbreviations must not result in new knowledge concerning the subject matter that could not have been proved without the definitions.

To make these notions more precise, consider a Secondary Expression, x , and a Possible Statement, s , such that x is not used to formulate s . We might say that s expresses some fact about the subject matter that is entirely expressible without x . Now suppose also that we have a subset

of the set of Theoretical Statements, A, such that x is not used to formulate any Statement in A, and A and $\bar{x}$ can be used to derive s. That is we are assuming that

$$A \cup \{\bar{x}\} \vdash s$$

holds. To say that $\bar{x}$ is non-creative is simply to assert that given these assumptions it follows that

$$A \vdash s$$

holds. That is, Statements not containing x derivable from $\bar{x}$ and other Statements not containing x, remain derivable from the set of Statements without $\bar{x}$. Whence, $\bar{x}$ does not deductively strengthen the theory by leading to new results not derivable without $\bar{x}$.

Formally, we have the following, and final, axiom.

Axiom 12. (Non-creativity) The Definition of any Secondary Expression, x, is formulated in such a way that any Possible Statement, s, not formulated in terms of x and derivable from $\bar{x}$ and a set of Theoretical Statements, A, no members of which are formulated in terms of x, remains derivable from A only.

Symbolically:

$$(\forall x \in SE) (\forall s \in PS) [\neg Fxs \rightarrow (\forall A \in P(TS)) (A \cup \{\bar{x}\} \vdash s \\ \& (\forall s_1 \in A) \neg Fxs_1 \rightarrow A \vdash s)]$$

It would be somewhat more accurate to say that Axioms 11 and 12 assert that definitional statements must satisfy weak eliminability and weak non-creativity. It is not entirely evident from these axioms that every Secondary Expression is actually eliminable from an axiomatic theory. Consider, for example, a definitional statement $\bar{x}$ and an arbitrary set of proved statements (theorems), definitions and other unproved statements (axioms) such that some of these elements are formulated in terms of x and some are not. It also may be the case that $\bar{x}$ is used in some of these proofs which means that $\bar{x}$ is used to derive some statements. If x is to be truly eliminable, then it must be shown that every single occurrence of x and $\bar{x}$ in this group is dispensable. In this sense of strong eliminability, not only must it be shown that equivalent definitions, theorems and axioms are forthcoming that are not formulated in terms of x ; but it also must be shown that the equivalent theorems can still be derived without $\bar{x}$.

We will now proceed to establish a theorem that expresses the expected result concerning the notion of strong eliminability. This idea is more complex than simply asserting that we can get by without a particular defined concept. Indeed, to remove one Secondary Expression is to alter the axiomatic theory completely. That is, if we eliminate a Secondary Expression, then we, in fact, construct a whole new axiomatization. Thus, we will need to be able to compare various axiomatizations of the same subject matter for their relative definitional, and relative derivational, strengths. To this end, we will introduce some preliminary definitions and results. We begin with a formal definition of an Axiomatically Structured Theory.

Definition 14. $\mathcal{A} = \langle L, PS, TS, PE, SE, F, \vdash, \models \rangle$ is an Axiomatically Structured Theory if and only if $\mathcal{A}$ satisfies Axioms 1 through 12.

Definition 15. The set of Definitions within an Axiomatically Structured Theory, $\mathcal{A}$, is defined as follows.

$$\text{Def}(\mathcal{A}) = \{s \in TS : (\exists x \in SE) s = \bar{x}\}.$$

Definition 16. The set of Theorems within an Axiomatically Structured Theory, $\mathcal{A}$, is defined as follows.

$$\text{Th}(\mathcal{A}) = \{s \in TS : (\exists A \in P(TS)) A \vdash s\}.$$

Definition 17. The set of Axioms within an Axiomatically Structured Theory, $\mathcal{A}$, is defined as follows.

$$\text{Ax}(\mathcal{A}) = TS \sim (\text{Def}(\mathcal{A}) \cup \text{Th}(\mathcal{A})).$$

For a moment let us alter our perspective slightly and apply the theory we are developing to our development itself. We are in the process of developing an axiomatic theory and, thus, what we say about the nature of axiomatics should apply to our own theoretical developments. The Statement labeled Definition 14 above seems to violate our assumptions concerning the "used in the formulation of" relationship. Definition 14 seems to be formulated in terms of eleven Theoretical Statements--Axioms 1-12. But we have used the "used in the formulation of" relationship in such a way that Fxs implies that x is either a Primary or Secondary Expression, and not a Statement at all. We consider Statements such as Definition 14 as being devices that simply minimize repetition of formulae. Definition 14 is to be considered as a "stand-in" for a

Statement of the following form.

Definition 14. $\mathcal{U} = \langle L, PS, TS, PE, SE, F, \vdash, \vDash \rangle$, is an Axiomatically Structured Theory if and only if $\mathcal{U}$ satisfies the following 12 conditions.

Axiom 1. PS is the set of well formed statements within some language L^1 , which is a modification of the underlying language L, effected as follows. . .

Axiom 2. $TS \subseteq PS$, TS is finite. . .

etc.

Axiom 12. $(\forall x \in SE)(\forall s \in PS)[\neg Fxs \rightarrow \dots]$.

Thus, in analyzing the F relationships in Definition 14 it is necessary to construct the full statement of the definition, and then to identify the F relationships within the statement.

In an axiomatization it is often the case that some theorems will be given without the proofs being displayed. This is especially true when an axiomatization of a subject matter is based on previously established theories. For example, an axiomatization of kinship relations may be based on Set Theory. In such a case a result from the previously established theory may be needed within the axiomatization. Such a result is usually stated without displaying the proof. In the kinship example, a useful theorem from Set Theory may be stated without its proof. On the other hand, Theorems are also listed often without proof when it is felt that the proof is trivial or straightforward. In such cases the proof is omitted since it is felt that the theorem can be verified easily.

Now, strictly speaking, if the proof is omitted, we cannot say from an observation of the axiomatization only, that the theorem is proved. That is, we cannot assert that the "is derived" ($\vdash$) relationship applies to the theorem. It seems that we can only make the weaker assertion that the statement "can be derived" ($\models$). Thus, it appears that such a theorem does not qualify for membership in $\text{Th}(\mathcal{U})$.

This dilemma is only apparent as the following argument is intended to establish. Within an axiomatization derived statements may be labeled theorems, propositions, corollaries, lemmas, etc. The position taken herein is that to label a statement in this way is to claim that the statement has in fact been derived even though the derivation may not be displayed. In fact, in practice, to label a statement in this way means that, if challenged, it is expected that a derivation will be forthcoming. That is, if a statement is labelled as a theorem, proposition, corollary or lemma, etc. then this is tantamount to asserting not only that the statement can be derived; but also that it has been derived and that the derivation will be supplied upon demand. As a label for those statements which have not been derived, but for which it is believed that they can be derived, we reserve the terms conjecture, thesis, etc. Thus, statements labeled as theorems, propositions, etc. are to be included within $\text{Th}(\mathcal{U})$ whether or not the proofs are displayed.

On the other hand conjectures, etc. cannot be said to be derived. Thus, they are to be excluded from $\text{Th}(\mathcal{U})$ as in Definition 16. If these statements are asserted within an axiomatization, then they must be

assumed to be true. That is, they function much like axioms. As appropriate, these statements will be classified in $Ax(\mathcal{U})$ as can be seen from Definition 17.

Finally, notice that $Def(\mathcal{U})$, $Th(\mathcal{U})$ and $Ax(\mathcal{U})$ partition TS. While omitting the proof of this result we point out that the derivation is straightforward from Definitions 15, 16 and 17, and Axiom 7.

Theorem 15. $Def(\mathcal{U})$, $Th(\mathcal{U})$, $Ax(\mathcal{U})$ partition TS where

$\mathcal{U} = \langle L, PS, TS, PE, SE, F, \vdash, \models \rangle$ is an
Axiomatically Structured Theory.

Theorem 15 points out another idiosyncrasy of axiomatics that can be discussed by applying, as before, the theory we are developing to the development underway. Theorem 15 is a component of an axiomatization, $\mathcal{U}$. The term partition appears in Theorem 15, but we have not provided a formal definition of this term. The definition of "partition" would be a statement such as the following.

Sets $A_1, \dots, A_n$ are said to partition a set B if and only if
for all i, j , $i, j = 1, \dots, n$, $A_i \cap A_j = \emptyset$ and $\bigcup_{i=1, n} A_i = B$.

We have assumed herein that the axiomatization we are engaged in is based on some formalization of Set Theory which includes a suitable definition of "partition".

Looking further at what has been developed so far we see that our definitions, axioms and theorems are teeming with symbols: $\forall, \exists, =, \emptyset, \varepsilon$,

$\rightarrow$, $\&$, $\vee$, etc. that have not be defined formally. Thus, we see that axiomatizations are usually based on other theories whose definitions, etc. may be brought into play freely. These theories are seen to be part of L , the underlying language. By limiting the number of proofs and definitions in this way, it is possible to focus on the conceptual framework of the subject matter under consideration. As with unproved theorems, however, to use an undefined concept is to assert implicitly that a definition will be found within the theories upon which the axiomatization is based.

Returning now to our axiomatic theory we introduce some notational conventions that will simplify subsequent developments. We will assume that $\mathcal{U}$, $\mathcal{U}^*$, $\mathcal{U}^{**}$ denote only Axiomatically Structured Theories where

$$\mathcal{U} = \langle L, PS, TS, PE, SE, F, \vdash, \vDash \rangle$$

$$\mathcal{U}^* = \langle L^*, PS^*, TS^*, PE^*, SE^*, F^*, \vdash^*, \vDash^* \rangle$$

$$\mathcal{U}^{**} = \langle L^{**}, PS^{**}, TS^{**}, PE^{**}, SE^{**}, F^{**}, \vdash^{**}, \vDash^{**} \rangle$$

To denote Definitions in these elements we write $\bar{x}$, $\bar{x}^*$ and $\bar{x}^{**}$ for the Definition of x in $\mathcal{U}$, $\mathcal{U}^*$ and $\mathcal{U}^{**}$, respectively.

The next definition introduces the notion of a Syntactic Simplification, $\mathcal{U}^*$, of a theory $\mathcal{U}$. A Syntactic Simplification will be constructed within the same language, L , on the basis of the same Primitive Expressions. Changing L would be more akin to a translation of $\mathcal{U}$. Altering PE, say by reducing PE, would result in a conceptual modification. The idea behind the concept of a Syntactical Simplification is that the set of Secondary Expression is reduced. In this way there are fewer theoretical terms and, hence, the number of theoretical components has been reduced.

This does not imply that the syntax of $\mathcal{Q}$ is simplified in the sense that the elements of TS^* are somehow simpler than the elements of TS . On the contrary, since Secondary Expressions often serve as abbreviations for longer expressions, the elements of TS^* can be expected to be syntactically more complex. Syntactic Simplification of $\mathcal{Q}$ does imply that from the point of view of $\mathcal{Q}$ we have simplified the syntax by eliminating Theoretical Expressions.

In effect a Syntactic Simplification will be a function mapping PS into PS^* . It might be felt that this function should map

$$PS \sim \{x : x \in SE \sim SE^*\}$$

into PS^* . But notice that the same sentence may be labeled as a Definition of an $x \in SE \sim SE^*$ and as a Theorem in $\mathcal{Q}$. Our intention is to use Axiom 11 to provide the images in PS^* of Statements in PS . Thus, when the same sentence receives labels of these two types we can use the same Statement in PS^* as the image of each Statement. In other words, we can assume, with no loss of generality, that the function maps PS entirely in PS^* .

In the Syntactic Simplification $\mathcal{Q}^*$ of $\mathcal{Q}$ we expect the content of the elements of TS to be reflected in corresponding elements of TS^* . That is, we will require a one-to-one correspondence (1-1 function) between TS^* and TS . The definitions of the deleted Secondary Expressions would be superfluous in TS^* and, hence, by their deletion, the correspondence will be from a subset of TS into TS^* . The image of elements under the 1-1 function will be the image of the element under the function from PS to PS^* . To preserve the content of elements in TS , we mean that if $x \in TS$, and

$x^* \in TS^*$ corresponds to x , then we will be able to prove that x is equivalent to x^* . This proof will be possible on the basis of elements in PS . That is, we expect that deleted definitions in TS will be required, in general, to establish the equivalence between x and x^* and, hence, the proof must be carried out in PS rather than PS^* .

We will also require that the image, under the 1-1 function, of a Definition in $\mathcal{U}$ will be the Definition of the same Expression in $\mathcal{U}^*$. Furthermore, images of Axioms of $\mathcal{U}$ will be Axioms of $\mathcal{U}^*$. These assumptions assure us that Theorems and Axioms in $\mathcal{U}$ correspond to Theorems and Axioms in $\mathcal{U}^*$, respectively. Without this result $\mathcal{U}^*$ would consist of a significant restructuring of $\mathcal{U}$ and not simply a simplification.

Our formal definition is as follows:

Definition 16. An Axiomatically Structured Theory $\mathcal{U}^*$ is a Syntactic

Simplification of an Axiomatically Structured Theory $\mathcal{U}$ if and only if the following conditions hold.

- (i) $PE^* = PE$
- (ii) $SE^* \subset SE$
- (iii) $L^* = L$ and L^{1*} arises from L^1 by deleting from L^1 all elements in $SE \setminus SE^*$ added to L when L^1 was formed; and by deleting all special transformation rules added to L concerning elements in $SE \setminus SE^*$.
- (iv) There is a function g from PS into PS^* .
- (v) There is a 1-1 function h from $TS \setminus \{\bar{x} : x \in SE \setminus SE^*\}$ into TS^* such that for all $s \in TS \setminus \{\bar{x} : x \in SE \setminus SE^*\}$ $h(s) = g(s)$.

- (vi) For all $x \in SE^*$, $h(\bar{x}) = \bar{x}^*$.
- (vii) For all $s \in Ax(\mathcal{Q}^*)$, $h^{-1}(s) \in Ax(\mathcal{Q})$.
- (viii) For all $s \in PS$, $PS \models s \leftrightarrow g(s)$.

Condition (iii) of this definition can be seen to imply that $PS^* \subset PS$. In forming L^{1*} we have simply removed any special Expressions and transformation rules for elements in $SE \sim SE^*$ that were added when L^1 was formed from L . Since everything expressible in PS^* results from L , PE and SE , everything expressible in PS^* can be expressed in L^1 , or $PS^* \subseteq PS$. Since at least one Secondary Expression is removed in going from $\mathcal{Q}$ to $\mathcal{Q}^*$ it follows that there are some things expressible in PS that are not expressible in PS^* . Whence, $PS^* \subset PS$.

We can also show that for all $s \in TS^*$ both

$$s \in Def(\mathcal{Q}^*) \leftrightarrow h^{-1}(s) \in [Def(\mathcal{Q}) \sim \{\bar{x} : x \in SE \sim SE^*\}]$$

and

$$s \in Th(\mathcal{Q}^*) \leftrightarrow h^{-1}(s) \in Th(\mathcal{Q})$$

where $\mathcal{Q}^*$ is a Syntactic Simplification of $\mathcal{Q}$. It is also clear that since $SE^* \subset SE$, Syntactic Simplification is an asymmetric relationship. As stated in Theorem 17 below, Syntactic Simplification is a transitive relationship. Before stating this theorem, we state as a theorem a useful result from logic not requiring a proof herein.

Theorem 16. Let A and B be sets and let s be a Statement. If $A \models s$ and

$A \subseteq B$, then $B \models s$.

Theorem 17. Syntactic Simplification is a transitive relationship.

To establish this theorem, let $\mathcal{U}^{**}$ be a Syntactic Simplification of $\mathcal{U}^*$ and let $\mathcal{U}^*$ be a Syntactic Simplification of $\mathcal{U}$. Conditions (i), (ii) and (iii) of Definition 16 obviously hold between $\mathcal{U}^{**}$ and $\mathcal{U}$. For condition (iv) let g_1 be the function from PS to PS*, and let g_2 be the function from PS* to PS**. The composite function: $g_1 \circ g_2(s) = g_2(g_1(s))$ will be a function from PS to PS**. Let h_1 and h_2 be the 1-1 functions postulated in condition (v) so that

$$h_1 : TS \sim \{\bar{x} : x \in SE \sim SE^*\} \rightarrow TS^*$$

$$h_2 : TS^* \sim \{\bar{x}^* : x \in SE^* \sim SE^{**}\} \rightarrow TS^{**}.$$

We will show that the composite function $h_1 \circ h_2(s) = h_2(h_1(s))$ will be a 1-1 function such that

$$h_1 \circ h_2 : TS \sim \{\bar{x} : x \in SE \sim SE^{**}\} \rightarrow TS^{**}.$$

To this end, let $scTS \sim \{\bar{x} : x \in SE \sim SE^{**}\}$.

Notice that

$$SE \sim SE^{**} = (SE \sim SE^*) \cup (SE^* \sim SE^{**}),$$

and so

$$scTS \sim \{\bar{x} : x \in SE \sim SE^*\}.$$

Therefore $h_1(s) \in TS^*$. We must show that $h_1(s)$ is in the domain of h_2 .

Suppose this is not the case, or,

$$h_1(s) = \bar{x}^*$$

where

$$x \in SE^* \sim SE^{**}.$$

We now have

$$s = h_1^{-1}(h_1(s)) = h_1^{-1}(\bar{x}^*) = \bar{x},$$

and, from the above equality for $SE \sim SE^{**}$,

$$x \in SE \sim SE^{**}.$$

This contradicts the choice of s .

Therefore, $s \in TS \sim \{\bar{x} : x \in SE \sim SE^{**}\}$ implies that $h_1(s)$ is in the domain of h_2 . It follows that the composite function $h_1 \circ h_2$ is a 1-1 function from $TS \sim \{\bar{x} : x \in SE \sim SE^{**}\}$ to TS^{**} as required in condition (v) of Definition 15. In addition

$$\begin{aligned} h_1 \circ h_2(s) &= h_2(h_1(s)) \\ &= h_2(g_1(s)) \\ &= g_2(g_1(s)) \\ &= g_1 \circ g_2(s) \end{aligned}$$

where $s \in TS \sim \{\bar{x} : x \in SE \sim SE^{**}\}$ and condition (v) is satisfied.

For condition (vi) we have

$$\begin{aligned} h_1 \circ h_2(\bar{x}) &= h_2(h_1(\bar{x})) \\ &= h_2(\bar{x}^*) \\ &= \bar{x}^{**} \\ &= \bar{x} \end{aligned}$$

as required. For condition (vii)

$$s \in \text{Ax}(\mathcal{U}^{**}) \leftrightarrow h_2^{-1}(s) \in \text{Ax}(\mathcal{U}^*)$$

$$\leftrightarrow h_1^{-1}(h_2^{-1}(s)) \in \text{Ax}(\mathcal{U})$$

$$\leftrightarrow (h_1 \circ h_2)^{-1}(s) \in \text{Ax}(\mathcal{U}).$$

Finally, condition (viii) follows since $\text{PS}^* \subseteq \text{PS}$; which means

$$\text{PS} \models s \leftrightarrow g_1(s)$$

and

$$\text{PS}^* \models g_1(s) \leftrightarrow g_2(g_1(s))$$

imply

$$\text{PS} \models s \leftrightarrow g_2(g_1(s));$$

or

$$\text{PS} \models s \leftrightarrow g_1 \circ g_2(s)$$

The next definition introduces the notion of a Syntactic Reduction of an Axiomatically Structured Theory. A Syntactic Reduction, $\mathcal{U}^*$, of $\mathcal{U}$ is a Syntactic Simplification that preserves the derivational strength of $\mathcal{U}$. That is, if a Statement can be derived in $\mathcal{U}$ from a subset of TS, then the image of the Statement can be derived in $\mathcal{U}^*$ from a subset of TS^* .

Definition 17. An Axiomatically Structured Theory $\mathcal{Q}^*$ is a Syntactic Reduction of an Axiomatically Structured Theory $\mathcal{Q}$ if and only if the following conditions are satisfied.

(i) $\mathcal{Q}^*$ is a Syntactic Simplification of $\mathcal{Q}$.

(ii) $(\forall s \in PS)[(\exists A \in P(TS)) A \models s$
 $\rightarrow (\exists A^* \in P(TS^*)) A^* \models^* g(s)]$

where g is the function from PS to PS^* .

It may be felt that if $\mathcal{Q}^*$ is a Syntactic Simplification of $\mathcal{Q}$, then it follows that $\mathcal{Q}^*$ is a Syntactic Reduction of $\mathcal{Q}$. But this implication does not hold as the following argument is intended to suggest where $\mathcal{Q}^*$ is a Syntactic Simplification of $\mathcal{Q}$. Suppose we have

$$A = \{s_1, \dots, s_n\}$$

and

$$A \models s$$

where $s \in PS, s_1 \in TS, \dots, s_n \in TS$. It might be argued that the images of $s_1, \dots, s_n$ can be used to prove $g(s)$ as required in Definition 17. But notice that, in general, all we can assert is

$$PS \models s_1 \leftrightarrow g(s_1),$$

$$\vdots$$

$$PS \models s_n \leftrightarrow g(s_n),$$

and

$$PS \models s \leftrightarrow g(s).$$

Together these statements imply (using Theorem 20 below)

$$PS \cup A^* \models g(s)$$

where $A^* = \{g(s_1), \dots, g(s_n)\}$. That is, we have no guarantee that the proof can be carried through in its entirety in $\mathcal{Q}^*$ alone. Thus, condition (ii) of Definition 17 does not necessarily follow from condition (i).

As will be seen below when we prove our theorem concerning strong eliminability, a Syntactic Simplification can be shown to be a Syntactic Reduction if the images of the elements in PS are chosen carefully enough. Indeed, by judicious selection of the images of elements in PS, i.e., using Axiom 11 and 12 in concert, we can establish that the Syntactic Simplification is also a Syntactic Reduction. The details of this process will be described in the proof of Theorem 21 below.

Syntactic Reduction is obviously an asymmetric relationship. We can also show that Syntactic Reduction is a transitive relationship.

Theorem 18. Syntactic Reduction is a transitive relationship.

To prove Theorem 18 let $\mathcal{Q}^{**}$ be a Syntactic Reduction of $\mathcal{Q}^*$ and let $\mathcal{Q}^*$ be a Syntactic Reduction of $\mathcal{Q}$. From Theorem 17 we know that $\mathcal{Q}^{**}$ is a Syntactic Simplification of $\mathcal{Q}$. For condition (ii) of Definition 17 we must prove for all $s \in PS$, if

$$(\exists A \in P(TS)) A \models s,$$

then

$$(\exists A^{**} \in P(TS^{**})) A^{**} \models^{**} g(s).$$

Let g_1, g_2 be the functions from PS to PS* and from PS* to PS**, respectively.

The composite function $g_1 \circ g_2(s) = g_2(g_1(s))$ will be a function from PS to PS**. Assume

$$(\exists A \in P(TS)) A \models s.$$

We know from Definition 17 applied to $\mathcal{U}$ and $\mathcal{U}^*$ that

$$(\exists A^* \in P(TS^*)) A^* \models^* g_1(s).$$

Applying Definition 17 to $\mathcal{U}^*$ and $\mathcal{U}^{**}$ implies

$$(\exists A^{**} \in P(TS^{**})) A^{**} \models^{**} g_2(g_1(s)),$$

as required.

The next definition introduces the notion of a Primitively Based Axiomatically Structured Theory. A theory will be Primitively Based just in case the theory does not utilize any Secondary Expressions. Thus, all statements in a Primitively Based theory are formulated in terms of Primitive Expressions only.

Definition 18. An Axiomatically Structured Theory, $\mathcal{U}$, is Primitively Based if and only if $SE = \emptyset$.

In practice we do not expect to encounter Primitively Based Axiomatically Structured Theories. Axiomatics is intended to proceed by way of definitions of concepts in order to simplify the theoretical constructions. In fact, if we consider some axiomatic development that is rich in definitions, we see that the theory may well be practically incomprehensible without defined concepts. Consider, for example, the set

of definitions that underly the proof of Gödel's theorem. To attempt to render these definitions in primitive terms only would not only be a staggering task, it would also be of questionable value. We expect, however, that this could be done, in principle. This is the sense of strong eliminability that definitions must satisfy. In Theorem 21 below we prove that Axiomatically Structured Theories meet this requirement.

The Proof of Theorem 21 will require 2 additional theorems from Logic that we state without proof.

Theorem 19. For any statements s, s_1 and any set A , if $A \cup \{s\} \models s_1$ and $A \models s$, then $A \models s_1$.

Theorem 20. For any statements $s_1, \dots, s_n, s$ and any set of statements A if $A \cup \{s_1, \dots, s_n\} \models s$, then $A \models s_1 \& \dots \& s_n \rightarrow s$, and conversely.

Theorem 21. Every Axiomatically Structured Theory is Syntactically Reducible to a Primitively Based Axiomatically Structured Theory.

To establish this theorem let $\mathcal{U} = \langle L, PS, SE, TS, F, \vdash, \models \rangle$ be an Axiomatically Structured Theory. In the event that $SE = \emptyset$, $\mathcal{U}$ is Primitively Based and there is nothing to prove. Assume $SE \neq \emptyset$.

Theorem 2 assures us that SE is finite and, thus, we can represent SE as follows.

$$SE = \{a_1, \dots, a_n\}$$

Consider an arbitrary member of SE , say a_1 . We will construct an Axiomatically Structured Theory $\mathcal{U}^*$ having as Secondary Expressions : $SE \cup \{a_1\}$. The construction will be carried out so that

$\mathcal{Q}^*$ is also a Syntactic Reduction of $\mathcal{Q}$. This method can be employed in turn to construct another Axiomatically Structured Theory $\mathcal{Q}^{**}$ from $\mathcal{Q}^*$. Since Syntactic Reduction is a transitive relationship, $\mathcal{Q}^{**}$ will be a Syntactic Reduction of $\mathcal{Q}$. The finiteness of SE assures us that eventually (after n applications of the method to be described) SE will be exhausted and the resulting Syntactic Reduction of $\mathcal{Q}$ will be Primitively Based.

As the starting point in the construction of $\mathcal{Q}^*$ we have the following three definitions.

$$L^* = L$$

$$PE^* = PE$$

$$SE^* = SE \sim \{a_i\}$$

Now L^{1*} is structured from a consideration of both $L^*(L)$ and L^1 . If a_i is in the lexicon of $L^*(L)$, a_i remains in the lexicon of L^{1*} . However, any changes in transformation rules concerning a_i introduced when L^1 was formed from L are restored to their original status. If a_i is not in the lexicon of L^* , then a_i was added to the lexicon of L to form the lexicon of L^1 . All requisite transformation rules for a_i were also added to L to form L^1 . In this case a_i , and its transformation rules, are deleted from L^1 to form L^{1*} . In either case, L^{1*} becomes an extension of L^* that satisfies axiom 1. Whence PS^* is the set of well formed statements within L^{1*} . Notice that for all $s \in PS^*$, $\exists Fa_i s$; also notice that $PS^* \subseteq PS$.

The relationship F will have the same meaning in $\mathcal{Q}^*$ as in $\mathcal{Q}$ and will not require special notation. That is, when Fxy appears, the elements x and y , along with the context, will determine whether F in $\mathcal{Q}$ or F in $\mathcal{Q}^*$ is being employed.

Next, we will define a function from PS to PS^* . Our intention is to use Axiom 11 in $\mathcal{U}$ to select an image for each element in PS . But notice that the element equivalent to a sentence forthcoming from Axiom 11 is unique up to equivalence only. Notice also that a sentence may occur twice in PS ---e.g., as both a definition and as a theorem. We must employ some method that assures us that these two occurrences of the same sentence are assigned identical images. We must first group sentences into equivalence classes under equality, and then define our function using Axiom 11 to select the same image for each member of each equivalence class.

As a definition for the required equivalence classes, for all $s \in PS$ let

$$[s] = \{s' : s' \in PS \text{ \& } s' = s\}.$$

Our function, g , from PS to PS^* is defined as follows for all $s \in PS$.

Case 1. If $\neg \exists a_1 s$ let $g(s) = s$.

Case 2. If $\exists a_1 s$ we know there is at least one $s' \in PS$ such that s' satisfies Axiom 11 in $\mathcal{U}$. Select one of these elements, s' . For all $s'' \in [s]$ let $g(s'') = s'$.

Notice that for all $s \in PS$, $\exists a_1 g(s)$ or $g(s) \in PS^*$. Whence, g is a function from PS into PS^* .

The function g will be used to define a 1-1 function from $TS \sim \{a_1\}$ into a subset (later to be designated as TS^*) of PS^* . The elements of $TS \sim \{a_1\}$ can be designated as $s_1, \dots, s_m$ by ordering them in some

distinct order. Axiom 2 in $\mathcal{U}$ assures us that they have distinct labels and, thus, we can relabel them as desired. We have

$$TS \sim \{\overline{a_i}\} = \langle s_1, \dots, s_m \rangle$$

which we use to define TS^* :

$$TS^* = \langle g(s_1), \dots, g(s_m) \rangle$$

or

$$TS^* = \langle s_1^*, \dots, s_m^* \rangle$$

where for $j=1, \dots, m$ $s_j^* = g(s_j)$. The subscripts on elements in TS^* can serve as the labels required in Axiom 2. It is important to remember that $TS \sim \{\overline{a_i}\}$, as written above, and TS^* are ordered sets. The same sentence may appear more than once in either of these sets but it is imperative that we do not delete one of these occurrences.

It should be clear from this construction that we have a 1-1 function, h , from $TS \sim \{\overline{a_i}\}$ to TS^* . Elements in SE^* must be associated with unique elements in TS^* : their respective definitions. Let f be the 1-1 function from SE to TS postulated in Axiom 5 for $\mathcal{U}$. For all elements x in SE^* , $\overline{x}$ denotes the definition of x in $\mathcal{U}$ and $\overline{x}^*$ denotes the definition of x in $\mathcal{U}^*$ where

$$\overline{x}^* = h(\overline{x}) = g(\overline{x}) = g(f(x)).$$

The composite function

$$f \circ h(s) = h(f(s))$$

defines a 1-1 function from SE^* into TS^* .

We will next define the relation $\vdash^*$. For all s^* in TS^* and for all $A^* \in P(TS^*)$. $A^* \vdash^* s^*$ if and only if there is an $A \in P(TS)$.

$$A \vdash h^{-1}(s^*) \text{ and } A^* = \{h(s) : s \in A \text{ \& } s \neq a_i\}.$$

In other words, $A^* \vdash^* s^*$ holds just in case s^* is the image of a proved element, s , in TS ; and A^* consists of the images of the statements in TS used to prove s .

Finally, we define $\models^*$ as $\models$ restricted to $P(PS^*) \times PS^*$. That is,

$$A \models^* s \text{ if and only if } A \models s, A \in P(PS^*) \text{ and } s \in PS^*.$$

We now must show that

$$\mathcal{N}^* = \langle L^*, PE^*, SE^*, PS^*, TS^*, F, \vdash^*, \models^* \rangle$$

is an Axiomatically Structured Theory.

The construction of L^{1*} resulting in PS^* has been carried out to assure us that Axiom 1 holds for $\mathcal{N}^*$. As pointed out above for all $s \in PS$, $g(s) \in PS^*$. Thus, $TS^* \subseteq PS^*$. The finiteness of TS assures us that TS^* is finite and we see that $\mathcal{N}^*$ satisfies Axiom 2. Axioms 3 and 4 obviously hold for $\mathcal{N}^*$; and we have described the composite function from SE^* into TS^* that satisfies Axiom 5: $f \circ h$.

To prove Axiom 6 consider an element $a_j \in SE^*$ where $j \neq 1$. From Axiom 6 in $\mathcal{N}$ we know that

$$Fa_j \bar{a}_j$$

must hold. There are two cases to consider. First, if $\neg Fa_j \bar{a}_j$, then

$$\bar{a}_j^* = g(\bar{a}_j) = \bar{a}_j,$$

and, thus,

$$Fa_j \bar{a}_j^*$$

as required in Axiom 6. Now suppose $Fa_i \bar{a}_j$. From the choice of the image of $\bar{a}_j, g(\bar{a}_j)$, we know that for all $w \in PEUSE$

$$Fwg(\bar{a}_j) \leftrightarrow w \neq a_i \text{ \& } (Fwa_j \vee Fwa_i)$$

Now $a_j \neq a_i$ and since we have $Fa_j \bar{a}_j$ we also have $Fa_j g(\bar{a}_j)$ or $Fa_j \bar{a}_j^*$.

Whence, Axiom 6 is satisfied by $\mathcal{U}^*$.

To prove Axiom 7 suppose there is an $A \in P(TS^*)$ and an element $a_j (j \neq i)$ in SE^* such that

$$A^* \vdash^* \bar{a}_j^*.$$

This means we have an A such that

$$A \vdash h^{-1}(\bar{a}_j^*),$$

or

$$A \vdash \bar{a}_j.$$

This contradicts Axiom 7 in $\mathcal{U}$. In other words, $\mathcal{U}^*$ satisfies Axiom 7.

To establish Axiom 8 we must show that there is no sequence of the form

$$Fxx_1^* \text{ \& } Fx_1x_2^* \text{ \& } \dots \text{ \& } Fx_{n-1}x_n^* \text{ \& } Fx_nx^*$$

where $x, x_1, \dots, x_n$ are all members of SE^* . Suppose such a sequence exists. We will show that such a sequence results in cyclic sequence of "used in the formulation of" relationships in $\mathcal{N}$, contradicting the fact that $\mathcal{N}$ satisfies Axiom 8.

Consider an arbitrary element, x_j , in this sequence. There are two possibilities to consider. First, suppose $\neg Fa_i \bar{x}_j$ holds. In this case $\bar{x}_j^* = \bar{x}_j$ and, thus, $Fx_{j-1} \bar{x}_j$ holds since we have $Fx_{j-1} \bar{x}_j^*$. On the other hand, suppose $Fa_i \bar{x}_j$ holds. From the choice of $\bar{x}_j^*$ we know for all $w \in PEUSE$

$$Fwx_j^* \leftrightarrow w \neq a_i \ \& \ (Fwx_j \vee Fwa_i).$$

Since $Fx_{j-1} \bar{x}_j^*$ we have

$$Fx_{j-1} \bar{x}_j \vee Fx_{j-1} \bar{a}_i.$$

Now consider the cyclic sequence of "F" relationships in $\mathcal{N}^*$ and consider the arbitrary portion of the sequence where we have

$$Fx_{j-1} \bar{x}_j^*.$$

Replace this portion of the sequence as follows.

If $\neg Fa_i \bar{x}_j$ the replacement becomes

$$Fx_{j-1} \bar{x}_j$$

which we have shown above to hold. If $Fa_i \bar{x}_j$ and if $Fx_{j-1} \bar{x}_j^*$, then again use as a replacement

$$Fx_{j-1} \bar{x}_j.$$

If $Fa_i \bar{x}_j$ and if $Fx_{j-1} \bar{a}_i$, then use as a replacement

$$Fx_{j-1} \bar{a}_i \ \& \ Fa_i \bar{x}_j.$$

That is, $Fx_{j-1} \bar{x}_j^*$ is replaced in any case by either

$$Fx_{j-1} \bar{x}_j$$

or

$$Fx_{j-1} \bar{a}_i \ \& \ Fa_i \bar{x}_j.$$

Since we can do this for each j such that $j=1, \dots, n$, we see that we have generated a cyclic sequence of "F" relationships within $\mathcal{V}$ from the cyclic sequence in $\mathcal{V}^*$. This result contradicts the fact that $\mathcal{V}$ satisfies Axiom 8. Therefore, $\mathcal{V}^*$ satisfies Axiom 8 also.

Deductive Dependency is defined in $\mathcal{V}^*$ from Γ^* in a manner analogous to the definition of Deductive Dependency in $\mathcal{V}$ from Γ . We must show that Deductive Dependency in $\mathcal{V}^*$ is asymmetric: Axiom 8. Suppose on the contrary that we have elements s^*, s_1^* such that s^* is Deductively Dependent on s_1^* and s_1^* is Deductively Dependent on s^* . This means we have elements

$$s_2^*, \dots, s_{p+q}^*$$

and sets

$$A_1^*, \dots, A_{p+q+1}^*$$

such that

$$\begin{array}{ccc}
 s_1^* \in A_1^* & \vdots & A_1^* \vdash^* s_2^* \\
 & & \\
 s_p^* \in A_p^* & & A_p^* \vdash^* s^* \\
 & & \\
 s_{p+1}^* \in A_{p+1}^* & & A_{p+1}^* \vdash^* s_{p+1}^* \\
 & & \\
 s_{p+1}^* \in A_{p+2}^* & \vdots & A_{p+2}^* \vdash^* s_{p+2}^* \\
 & & \\
 s_{p+q}^* \in A_{p+q+1}^* & & A_{p+q+1}^* \vdash^* s_1^*
 \end{array}$$

The relation $\vdash^*$ is defined between $P(TS^*)$ and TS^* which means we can speak of the inverse of the elements $s^*, s_1^*, \dots, s_{p+q}^*$. Let $g^{-1}(s^*)=s$ and $g^{-1}(s_j^*)=s_j$ for $j=1, \dots, p+q$. Applying the definition of $\vdash^*$ for each set A_j^* ($j=p+q+1$) we have sets A_j where

$$\begin{array}{ccc}
 s_1 \in A_1 & \vdots & A_1 \vdash s_2 \\
 & & \\
 s_p \in A_p & & A_p \vdash s \\
 & & \\
 s_{p+1} \in A_{p+1} & & A_{p+1} \vdash s_{p+1} \\
 & & \\
 s_{p+1} \in A_{p+2} & \vdots & A_{p+2} \vdash s_{p+2} \\
 & & \\
 s_{p+q} \in A_{p+q+1} & & A_{p+q+1} \vdash s_1
 \end{array}$$

But now we have shown that in $\mathcal{Q}$ s is Deductively Dependent on s_1 and s_1 is Deductively Dependent on s . That is, our original assumption leads to a contradiction of Axiom 9 in $\mathcal{Q}$; rendering Axiom 9 valid in $\mathcal{Q}^*$.

Before proceeding to the proof of Axiom 10, we will state and prove an assertion that will be of general use in the remainder of the proof of this theorem.

"For all $s \in PS$, $\{\bar{a}_1\} \models s \leftrightarrow g(s)$ ".

To establish this fact, notice that if $s \in PS$ and $\bigwedge Fa_1s$ holds then $g(s)=s$. Obviously $\{\bar{a}_1\} \models s \leftrightarrow g(s)$ holds in this case. In the case that Fa_1s holds, then $\{\bar{a}_1\} \models s \leftrightarrow g(s)$ holds as well due to the use of Axiom 11 in $\mathcal{Q}$ in choosing $g(s)$.

To prove Axiom 10 assume $A^* \vdash^* s^*$ for $A^* \in P(TS^*)$ and $s^* \in TS^*$. Let $g(s)=s^*$ and let

$$A^* = \{h(s_1), \dots, h(s_p)\}$$

for $s_1, \dots, s_p \in TS$. From the definition of $\vdash^*$ we have an A such that $A \vdash s$. Axiom 10 in A implies $A \models s$. Now A consists, at least, of $\{s_1, \dots, s_p\}$ and may also include $\{\bar{a}_1\}$. In any event, we have $A \cup \{\bar{a}_1\} \models s$. Theorem 20 implies

$$\{\bar{a}_1\} \models s_1 \& \dots \& s_p \rightarrow s.$$

We also know (recalling that for $j=1, \dots, p$, $h(s_j)=g(s_j)$)

$$\{\bar{a}_1\} \models s_1 \leftrightarrow g(s_1)$$

$$\vdots$$

$$\{\bar{a}_1\} \models s_p \leftrightarrow g(s_p)$$

$$\{\bar{a}_1\} \models s \leftrightarrow g(s).$$

Combining these statements yields

$$\{\bar{a}_i\} F g(s_1) \ \& \ \dots \ \& \ g(s_p) \rightarrow s^*,$$

or

$$A^* \cup \{\bar{a}_i\} F s^*.$$

Now $\neg Fa_i s^*$ and for all $x \in A^*$ $\neg Fa_i x^*$. Thus, we can apply Axiom 12 in $\mathcal{U}$ to derive

$$A^* F s^*.$$

Whence, by the definition of F^* we have, as required

$$A^* F^* s^*.$$

To establish the validity of Axiom 11, let a_j be a member of SE^* and let s^* be a member of PS^* where $Fa_j s^*$. There are two cases to consider.

Case 1. $\neg Fa_i \bar{a}_j$

Now $a_j \in SE$ and $s^* \in PS$, so we can apply Axiom 11 in $\mathcal{U}$ to get an $s \in PS$ such that for all $w \in PEUSE$

$$(a) \quad Fws \leftrightarrow w \neq a_j \ \& \ (Fws^* \vee Fw\bar{a}_j)$$

and

$$\{\bar{a}_i\} F s \leftrightarrow s^*.$$

We will prove $s \in PS^*$. Suppose to the contrary that $s \notin PS^*$. This means

$Fa_i s$ holds. We now have from (a)

$$Fa_j s^* \vee Fa_i \bar{a}_j.$$

But $s^* \in PS^*$ implies $\neg Fa_i s^*$. The assumption for this case eliminates the other disjunction, and, thus, we know that $s \in PS^*$. In the case under consideration $\bar{a}_j^* = \bar{a}_j$ which means that (a) can be rewritten as

$$Fws \leftrightarrow w \neq a_j \ \& \ (Fws^* \vee Fw\bar{a}_j^*)$$

Also $PE^* = PE$ and $SE^* \subseteq SE$, which implies that this last expression holds for all $w \in PE^*USE^*$. We have chosen a_j arbitrarily and so we have proved

$$\begin{aligned} & (\forall x \in SE^*) (\forall s^* \in PS^*) [Fxs^* \\ & \rightarrow (\exists s \in PS^*) (\forall w \in PE^*USE^*) \\ & (Fws \leftrightarrow w \neq x \ \& \ (Fws^* \vee Fw\bar{x}^*))] \end{aligned}$$

which is the first part of Axiom 11 for $\mathcal{M}^*$. It remains to be shown that

$$\{\bar{a}_i^*\} \models^* s \leftrightarrow s^*.$$

By the choice of s we know

$$\{\bar{a}_i\} \models s \leftrightarrow s^*.$$

Notice that $\bar{a}_j = \bar{a}_j^*$ implying

$$\{\bar{a}_j^*\} \models s \leftrightarrow s^*.$$

Since $s \in PS^*$ and $s^* \in PS^*$ we also have

$$\neg Fa_i (s \leftrightarrow s^*).$$

Thus, " $s \leftrightarrow s^*$ " $\in PS^*$. Also $\{\bar{a}_j^*\} \in P(PS^*)$ and thus, as required, we have proved that

$$\{\bar{a}_j^*\} \models^* s \leftrightarrow s^*.$$

Case 2. $Fa_1 \bar{a}_j$.

This case is the more difficult of the two and a brief indication of the form of the proof is in order. We have s^* in PS^* such that $Fa_j s^*$. We first remove a_j from s^* using Axiom 11 in $\mathcal{U}$ (as in Case 1). As a result we will have an equivalent statement, s , that will contain a_1 . We now apply Axiom 11 again on a_1 to get another statement that contains neither a_1 nor a_j . This second statement will be equivalent to s^* and will be the statement in PS^* required for the validity of Axiom 11 in $\mathcal{U}^*$.

Turning now to the details of the proof, for the first step we apply Axiom 11 in $\mathcal{U}$ to a_j and s^* giving us s where for all $u \in PEUSE$

$$(b) \quad Fus \leftrightarrow u \neq a_j \ \& \ (Fus^* \vee F\bar{u}a_j).$$

Clearly $s \in PS$. Also $a_1 \neq a_j$ and $Fa_1 \bar{a}_j$ by assumption. That is, $Fa_1 s$ holds. We now apply Axiom 1 in $\mathcal{U}$ to a_1 and s giving us s_1 where for all $v \in PEUSE$

$$(c) \quad Fvs_1 \leftrightarrow v \neq a_1 \ \& \ a_1 \ \& \ (Fvs \vee F\bar{v}a_1).$$

Notice that $s_1 \in PS^*$ as required for Axiom 11 in $\mathcal{U}^*$.

We will prove that for all $w \in PE^*USE^*$

$$Fws_1 \leftrightarrow w \neq a_j \ \& \ (Fws^* \vee F\bar{w}a_j^*).$$

Assume first that Fws_1 holds for $w \in PE^*USE^*$. Applying (c) we have

$$w \neq a_1$$

and

$$Fws \vee F\bar{w}a_1.$$

Consider the left disjunct in this disjunction: Fws . Applying (b) yields

$$(d) \quad w \neq a_j \ \& \ (Fws \vee Fw\bar{a}_j).$$

Recall that $\bar{a}_j^*$ was chosen so that for all $x \in PEUSE$

$$(e) \quad Fx\bar{a}_j^* \leftrightarrow x \neq a_j \ \& \ (Fx\bar{a}_j \vee Fx\bar{a}_1).$$

Now $w \neq a_1$, and if $Fw\bar{a}_j$, then we have from (e) $Fw\bar{a}_j^*$. Thus, (d) implies

$$w \neq a_j \ \& \ (Fws \vee Fw\bar{a}_j^*)$$

Now assume the right disjunct from the above disjunction: $Fw\bar{a}_1$.

From (e) we know $Fw\bar{a}_j^*$. We must show that $w \neq a_j$. Assume to the contrary that $w = a_j$. This means $Fa_j s_1$ by assumption. From (c) we have either $Fa_j s$ or $Fa_j \bar{a}_1$. But $Fa_j s$ implies by (b) that $a_j \neq a_j$. On the other hand $Fa_j \bar{a}_1$ implies that $\bar{a}_j$ is Syntactically Dependent on a_j , which contradicts Theorem 3. Therefore, we conclude that Fws_1 implies $w \neq a_j$. From $Fw\bar{a}_j^*$ we have $Fws \vee Fw\bar{a}_j^*$, and thus again

$$w \neq a_j \ \& \ (Fws \vee Fw\bar{a}_j^*)$$

Whence we have shown

$$Fws_1 \rightarrow w \neq a_j \ \& \ (Fws \vee Fw\bar{a}_j^*).$$

To complete this portion of the proof, we must show that the converse of this statement holds.

There are two possibilities with the first being that $w \neq a_j$ & Fws^* holds. From (b) we have Fws . Now $s^* \in PS^*$ so Fws^* also implies $w \neq a_1$. From (c) we have Fws_1 as required.

The other possibility is that $w \neq a_j$ & $Fw\bar{a}_j^*$ holds. From (e) we know $w \neq a_1$, and

$$Fw\bar{a}_j \vee Fw\bar{a}_1.$$

The left disjunct, $Fw\bar{a}_j$, with $w \neq a_j$ implies by (b) that Fws holds. But $w \neq a_1$ so from (c) Fws_1 . The right disjunct, $Fw\bar{a}_1$, with $w \neq a_1$ implies also with (c) that Fws_1 holds.

Whence we have proved that

$$Fws_1 \leftrightarrow w \neq a_j \text{ \& \& } (Fws^* \vee Fw\bar{a}_j^*).$$

To complete the proof of the validity of Axiom 1₁ we must show

$$\{\bar{a}_j^*\} \vDash s^* \leftrightarrow s_1.$$

To begin, we have

$$\{\bar{a}_j\} \vDash s^* \leftrightarrow s,$$

which implies

$$\emptyset \vDash \bar{a}_j \rightarrow (s^* \leftrightarrow s).$$

We also know

$$\{\bar{a}_1\} \vDash \bar{a}_j \leftrightarrow \bar{a}_j^*$$

which implies

$$\{\bar{a}_i\} \models \bar{a}_j^* \rightarrow (s^* \leftrightarrow s),$$

or

$$\{\bar{a}_j^*\} \cup \{\bar{a}_i\} \models s^* \leftrightarrow s.$$

But

$$\{\bar{a}_i\} \models s \leftrightarrow s_1$$

which yields

$$\{\bar{a}_j^*\} \cup \{\bar{a}_i\} \models s^* \leftrightarrow s_1.$$

By assumption $s^* \in PS^*$ and from the choice of s_1 $\neg Fa_1 s_1$. That is,

$$\neg Fa_1 (s^* \leftrightarrow s_1)$$

and since $\neg Fa_i \bar{a}_j^*$ we can apply Axiom 12 in $\mathcal{U}$ to conclude that

$$\{\bar{a}_j^*\} \models s^* \leftrightarrow s_1.$$

Now $\bar{a}_j^* \in PS^*$ and $s^* \leftrightarrow s_1 \in PS^*$, and, thus,

$$\{\bar{a}_j^*\} \models^* s^* \leftrightarrow s_1.$$

This completes the proof of the validity of Axiom 11 from $\mathcal{U}^*$ and we must now prove that $\mathcal{U}^*$ satisfies Axiom 12. To this end consider an element $a_j \in SE^*$ and an $s^* \in PS^*$ such that $\neg Fa_j s^*$. Assume also that

$$A^* \cup \{\bar{a}_j^*\} \models^* s^*$$

where $A^* \in P(TS^*)$, and for all $s \in A^*$, $\neg Fa_j s$ holds. From the definition of $\bar{F}$ we know

$$(f) \quad A^* \cup \{\bar{a}_j^*\} \bar{F} s^*.$$

There are two cases to consider.

Case 1. $\neg Fa_i \bar{a}_j$.

In this case $\bar{a}_j^* = \bar{a}_j$ and thus we have

$$A^* \cup \{\bar{a}_j\} \bar{F} s^*.$$

Now applying Axiom 12 in $\mathcal{Q}$ to a_j yields

$$A^* \bar{F} s^*$$

and thus

$$A^* \bar{F}^* s^*$$

as required for Axiom 12 in $\mathcal{Q}^*$.

Case 2. $Fa_i \bar{a}_j$.

From the choice of $\bar{a}_j^*$ we know

$$\{\bar{a}_i\} \bar{F} \bar{a}_j^* \leftrightarrow \bar{a}_j.$$

Now (f) implies

$$A^* \bar{F} \bar{a}_j^* \rightarrow s^*.$$

Combining these results yields

$$A^* \cup \{\bar{a}_i\} \models \bar{a}_j \rightarrow s^*$$

or

$$A^* \cup \{\bar{a}_i\} \cup \{\bar{a}_j\} \models s^*.$$

In this case $\neg Fa_j \bar{a}_i$ since $Fa_j \bar{a}_i$ implies that $\bar{a}_i$ is Syntactically Dependent on a_i which is contradictory. Whence, applying Axiom 12 in $\mathcal{U}$ to a_j we have

$$A^* \cup \{\bar{a}_i\} \models s^*.$$

Now a_i does not appear in any $s \in A^*$, nor does a_i appear in s^* . Applying Axiom 12 in $\mathcal{U}$ to a_i yields

$$A^* \models s^*$$

or

$$A^* \models^* s^*.$$

Thus, $\mathcal{U}^*$ satisfies Axiom 12.

We have proved at this point that $\mathcal{U}^*$ is an Axiomatically Structured Theory. It remains to be shown that $\mathcal{U}^*$ is a Syntactic Reduction of $\mathcal{U}$.

Conditions (i), (ii) and (iii) of Definition 16 clearly hold between $\mathcal{U}^*$ and $\mathcal{U}$. The function g defined above maps PS into PS* as required in condition (iv). The function h satisfies condition (v), with condition (vi) also being satisfied by construction. Since condition (viii) holds by the choice of $g(s)$ for $s \in PS$ we need only be concerned with proving condition (vii).

To this end assume $s^* \in Ax(\mathcal{U}^*)$. This means $s^* \notin Dif(\mathcal{U}^*)$ and $s^* \notin Th(\mathcal{U}^*)$. Let $h^{-1}(s^*) = s$ and notice that since $s^* \notin Th(\mathcal{U}^*)$ there is no A such that $A \vdash s$. That is, $h^{-1}(s^*) \notin Th(\mathcal{U})$. If $s \in Def(\mathcal{U})$ then $s = \bar{x}$ for $x \in SE^*$. In this case

$$s^* = h(s) = h(\bar{x}) = \bar{x}^*$$

or $s^* \in Def(\mathcal{U}^*)$ which is contradictory. Therefore, $h^{-1}(s^*) \notin Def(\mathcal{U})$ and $h^{-1}(s^*) \notin Th(\mathcal{U})$ or $h^{-1}(s^*) \in Ax(\mathcal{U})$ as required.

Now suppose $h^{-1}(s^*) \in Ax(\mathcal{U})$. Therefore, there is no A such that $A \vdash h^{-1}(s^*)$ or there is no A^* such that $A^* \vdash^* s^*$. This means $s^* \notin Th(\mathcal{U}^*)$. Suppose $s^* \in Def(\mathcal{U}^*)$ or $s^* = \bar{x}^*$ for some $x \in SE^*$. This means

$$h^{-1}(s^*) = h^{-1}(\bar{x}^*) = \bar{x}$$

or $h^{-1}(s^*) \in Def(\mathcal{U})$ which is contradictory. Thus, $s^* \in Ax(\mathcal{U}^*)$ and we see that condition (vii) is satisfied by $\mathcal{U}^*$ and $\mathcal{U}$.

We have shown that $\mathcal{U}^*$ is a Syntactic Simplification of $\mathcal{U}$. It remains to be shown that $\mathcal{U}^*$ is a Syntactic Reduction of $\mathcal{U}$.

To prove that $\mathcal{U}^*$ is a Syntactic Reduction of $\mathcal{U}$ we must show that if there is an $A \in P(TS)$ such that

$$A \models s,$$

then there is an $A^* \in P(TS^*)$ such that

$$A^* \models^* g(s)$$

where $s \in PS$. Let s be in PS and let $A = \{a_1, \dots, a_p\}$ be such that $A \in P(TS)$

and

$$A \models s.$$

We will consider two cases

Case 1. $\bar{a}_i \notin A$.

In this case we have

$$\emptyset \models a_1 \ \& \ \dots \ \& \ a_p \rightarrow s$$

$$\{a_i\} \models a_1 \leftrightarrow h(a_1)$$

$$\{a_i\} \models a_p \leftrightarrow h(a_p)$$

$$\{a_i\} \models s \leftrightarrow g(s)$$

or

$$\{a_i\} \models h(a_1) \ \& \ \dots \ \& \ h(a_p) \rightarrow g(s).$$

Taking $A^* = \{h(a_1), \dots, h(a_p)\}$ we see that $A^* \in P(TS^*)$ and

$$A^* \cup \{\bar{a}_i\} \models g(s).$$

But a_i does not appear in any element in A^* nor in $g(s)$. Applying Axiom 12 in $\mathcal{U}$ yields

$$A^* \models g(s)$$

and, thus, by the definition of $\models^*$

$$A^* \models^* g(s).$$

Case 2. $\bar{a}_1 \in A$.

In this case we begin with

$$\{\bar{a}_1\} \vdash a_1 \& \dots \& a_p \rightarrow s.$$

The rest of this case proceeds exactly as in Case 1 leading to the same conclusion.

Finally, we have shown that $\mathcal{U}^*$ is a Syntactic Reduction of $\mathcal{U}$. We can apply the above method to SE^* eliminating, say a_j for $j \neq 1$ from SE . That is, SE^{**} is defined as follows.

$$\begin{aligned} SE^{**} &= SE \sim \{a_j\} \\ &= SE \sim \{a_i, a_j\}. \end{aligned}$$

Proceeding analogous to the above constructions yields $\mathcal{U}^{**}$ a Syntactic Reduction of $\mathcal{U}^*$. From the transitivity of Syntactic Reduction we know that $\mathcal{U}^{**}$ is a Syntactic Reduction of $\mathcal{U}$. Continuing in this way, eventually SE is exhausted, since SE is finite, leaving a Syntactic Reduction of $\mathcal{U}$ in which there are no Secondary Expressions. As a result, $\mathcal{U}$ has been Syntactically Reduced to a Primitively Based Axiomatically Structured Theory, QED.

Two other notions related to the concepts of Syntactic Simplification and Syntactic Reduction are of interest but will not be investigated in detail herein. We will indicate roughly how Definitions 16 and 17 can be altered to develop these concepts. First, consider Definition 16 and suppose $L \not\equiv L^*$ where $\mathcal{U}$ and $\mathcal{U}^*$ are Axiomatically Structured Theories.

Suppose also that there is a 1-1 correspondence between PE and PE* and between SE and SE*. Conditions (iv), (v), (vi) and (vii) of Definition 16 need not be modified, and the only modification that seems to be needed in condition (viii) is that PS must be replaced by PSUPS*. Condition (ii) of Definition 17 also can remain unchanged leading to the notion of a "reduction" of $\mathcal{U}$ into a totally new language. It would be more appropriate to refer to this process as a Translation of $\mathcal{U}$ into $\mathcal{U}^*$. A Translation of this type can be seen to be irreflexive, symmetric and transitive.

Another interesting notion arises if in Definition 16, $L^*=L$ but $PE^* \subset PE$. Suppose in this case $SE^*=SE$, condition (v) is altered to permit more statements (viz., definitions) in TS^* : h is 1-1 from TS into TS^* but not into TS . The remaining conditions are substantially unchanged. A simplification in this sense leads to an Axiomatically Structured Theory which is based on fewer primitive concepts but which retains the obtained results in the original theory. This might be called a Conceptual Simplification leading to a Conceptual Reduction. This relationship is irreflexive, asymmetric and transitive.

The classical notion of independent primitive concepts can be stated in terms of the notions of Conceptual Simplification and Conceptual Reduction. The primitive concepts within an Axiomatically Structured Theory are said to be independent if and only if there is no Conceptual Simplification for the theory. That is, primitive concepts are independent just in case it is not possible to introduce definitions that eliminate at least one of them, while preserving the derivational power of the theory.

This is the intended meaning behind the notions of Conceptual Simplification and Reduction.

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APPENDIX II

Matrices for the Photography Example
From Chapter I

Figure 1 – Adjacency Matrix for Syntactic Dependency A(SD)

[illegible]

Figure 2 – Adjacency Matrix for Deductive Dependency A(DD)

[illegible]

4

89

90

Figure 5 – A(D)³

[illegible]

Figure 6 A(D)⁴[illegible]

Figure 7 – A(D)⁵[illegible]

Figure 8 – A(D) ⁶

[illegible]

Figure 9 – A(D)⁷

[illegible]

Figure 10 - A(D)⁸

[illegible]

Figure 11 – A(D)⁹97

Figure 12 – A(D)¹⁰

[illegible]

Figure 13 – A(D)¹¹

[illegible]

Figure 14 – A(D)¹²

[illegible]

Figure 15 – A(D)¹³

[illegible]

Figure 16 – A(D)¹⁴

[illegible]

Figure 17 – A(D)¹⁵

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Figure 18 — The Complete Dependency Matrix $D = [\zeta_{1,15} A(D)^i]_{\#}$

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